Radial solutions of semilinear elliptic equations with prescribed asymptotic behavior

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Abstract
We consider semilinear elliptic equations of the form $\Delta u + g(u) = 0$ on $\mathbb{R}^N$. Given a suitable positive root $z$ of $g$, we discuss the existence of positive radial solutions $u$ with $u(\infty) = z$.

KEYWORDS
asymptotic behavior, semilinear elliptic equations

1 | INTRODUCTION

Let $N \geq 2$. We discuss the existence of radial solutions $u$ of the problem

$$\Delta u + g(u) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

with a prescribed limit $z = u(\infty) > 0$ satisfying $g(z) = 0$. Radial solutions of (1.1) have been studied in connection with many questions of mathematical physics like the existence of solitary waves and the nonlinear field equations [1, 2, 11]. The idea of having symmetric solutions for semilinear elliptic equations on general symmetric domains is now well understood by the moving plane method (see [3, 6]). If, for example, we consider a ball $B = B_R(0)$, a suitable nonlinearity $f$ and a sufficiently regular positive solution $u$ of

$$\begin{cases}
\Delta u + f(u) = 0 & \text{in} \quad B, \\
u = 0 & \text{on} \quad \partial B,
\end{cases}$$

then $u$ is radially symmetric $u = u(r = |x|)$ and $u'(r) \leq 0$. The proof of this result has the maximum principle for elliptic equations as its principal ingredient. When the radius $R \to \infty$, it then becomes natural to ask about the existence of radially symmetric solutions of (1.1) that vanish at infinity. This question attracted a lot of attention (see [1, 2, 4, 5, 8, 10] and the references therein) where variational and topological methods have been used in order to establish the existence of positive radial solutions of (1.1) with $u(\infty) = 0$. However, and up to our knowledge, none of the aforementioned works investigates the case with a prescribed limit different from 0.

The aim of this paper is to provide an ODE-based proof for the existence of positive radial solutions of (1.1) with a prescribed nonzero limiting behavior related to the roots of the nonlinearity $g$. This approach combines a type of a shooting argument together with the resolution of (1.2) with a particular $f$ (see Remark 1.5), and for sufficiently large $R > 0$. The radial solution $u = u(r)$ of (1.1) satisfies

$$\begin{cases}
-u'' - \frac{N-1}{r} u' = g(u) & \text{for} \quad 0 < r < \infty, \\
u(0) = \xi, \quad u'(0) = 0,
\end{cases}$$

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where $\xi$ has to be determined so that
\[
\lim_{r \to \infty} u(r) = z, \tag{1.4}
\]
with $z > 0$ being a prescribed root of $g$ which we will make precise hereafter. Let us describe our conditions on $g$. We assume that $g : [0, \infty] \to \mathbb{R}$ is a locally Lipschitz function, $g(0) = 0$ and $g(z) = 0$ for some $z > 0$. We also assume the following.

There exists $a \in ]0, z[$ such that $g(u) < 0$ for $0 < u < a$ and $g(u) > 0$ for $a < u < z$. \hspace{1cm} \tag{1.5}

There exists $0 < \tau < z$ such that $G(\tau) > G(z)$, where $G(t) = \int_0^t g(s) \, ds$. \hspace{1cm} \tag{1.6}

Having (1.6), we define
\[
\tau_0 = \sup \{ \tau > 0; \ G(\tau) > G(z) \} \tag{1.7}
\]
then, in view of (1.5) and (1.6), $\tau_0$ exists in $\mathbb{R}$ and $\tau_0 < a$. Moreover, we have
\[
G(z) \geq G(\tau) \quad \text{for every} \quad \tau \in [\tau_0, a]. \tag{1.8}
\]

We finally assume
\[
\liminf_{s \to u} \frac{g(s)}{s - a} > 0. \tag{1.9}
\]

We prove the following existence result.

**Theorem 1.1.** Let $g$ be a locally Lipschitz continuous function on $\mathbb{R}_+ = [0, \infty]$ such that $g(0) = 0$. Assume moreover that $z > 0$ is a positive root of $g$ with the conditions (1.5), (1.6) and (1.9) all satisfied. Then there exists $\xi \in ]0, \tau_0[$ and a solution $u \in C^2(\mathbb{R}_+)$ of (1.3) and (1.4), with $u(r) < z$ for $r \in [0, \infty[$ and $u'(r) > 0$ for $r \in ]0, \infty[$.

**Remark 1.2.** The assumptions of Theorem 1.1 are satisfied for $g(s) = s(s - z)(\alpha - s)$ with $0 < \frac{z}{2} < \alpha < z$.

**Remark 1.3.** The short time existence of (1.3) follows from a classical fixed point argument associated to the integral equation
\[
u(r) = \xi - \int_0^r s^{-(N-1)} \int_0^s \sigma^{N-1} g(u(\sigma)) \, d\sigma \, ds, \tag{1.10}
\]
where we extend $g$ to $\mathbb{R}$ by setting $g(s) = 0$ if $s < 0$. Indeed, since we are interested in solutions $0 < u < z$, and since $g(z) = 0$, we will furthermore adjust the extended $g$ and replace it with
\[
\tilde{g} = \begin{cases} 
  g & \text{on} \ [0, z[,
  0 & \text{elsewhere}.
\end{cases}
\]

However, in all what follows and for the sake of clarity, we always keep the notation $g$ for our nonlinearity.

**Remark 1.4.** For $r > 0$, Equation (1.10) is no longer singular, so that the solution $u$ can be extended, by the usual method, to be defined on a maximal interval $]0, r_\xi[$ upon which it satisfies the blow up alternative. In this paper, we shall prove that $r_\xi = \infty$ for all $\xi \in ]0, a[$.

**Remark 1.5.** The result of Theorem 1.1 is independent of any assumption on the dimension $N$. The main reason for that is related, as it was previously mentioned above, to solving (1.2) which is, actually, the only place in the proof where questions about the dimension arise. As a matter of fact, as $g(0) = 0$ then, for $f(s) = -g(z - s)$, the Dirichlet problem (1.2) admits a solution (see for instance [1, 9]) for sufficiently large $R$ without any kind of conditions on $N$.

Let us finally mention that a similar problem of existence of solutions, with a prescribed limit, for semilinear elliptic problems over a quarter-plane has been treated in [7]. In this work, the solution is not necessarily radial and the existence is based on Perron’s method after constructing suitable sub- and super-solutions.
2 | PROOF OF MAIN RESULT

The proof will follow from several lemmas. As a first step we observe that for small values of $r$, namely on a maximal interval $[0, r_\varepsilon]$, the existence of (1.3) is ensured by classical arguments (see Remarks 1.3 and 1.4). The first lemma shows the existence of $u(r)$ for all $r \geq 0$, whenever $u(0) = \xi \in [0, a]$, i.e. $r_\varepsilon = \infty$ for any $\xi \in [0, a]$. We begin by noting that, upon multiplying (1.3) by $u'$ and integrating between 0 and $r$, the following fundamental equality

$$G(u(r)) + \frac{1}{2}(u'(r))^2 + (N-1) \int_0^r \frac{(u'(s))^2}{s} ds = G(\xi)$$

(2.1)

is satisfied as long as $u$ exists. We have

Lemma 2.1 (Existence on $\mathbb{R}$). If $\xi \in [0, a]$, then (1.3) admits a unique solution $u \in C^2(\mathbb{R}_+)$.\[Proof.\] Let $\xi \in [0, a]$. First of all, it is worth mentioning that, as $g(\xi) < 0$ (see (1.5)), the solution $u$ will be increasing in a small neighborhood of 0, thus $u(r) > u(0) = \xi$ there. This is a consequence of the equation $u''(0) + g(\xi) = 0$ that results from (1.3) computed at $r = 0$. Since $u$ satisfies the blow up alternative, we need to show that $u$ and $u'$ are bounded on for $r$ bounded. However, thanks to (2.1), it will be enough only to show that $u$ remains bounded. We claim that

$$u \geq \xi \quad \text{for all} \quad r \geq 0.$$ \[Indeed, if this is not true, we set\]

$$r_1 = \sup\{r \geq 0; u(r) \geq \xi\},$$

then $0 < r_1 < r_\varepsilon$ and $u(r_1) = \xi$. From the definition of $r_1$ and by the continuity of $u$, we may then find $r_2$ such that $r_1 < r_2 < r_\varepsilon$ with

$$0 < u(r_2) < \xi = u(r_1).$$

By (1.5), we observe that $G$ is strictly decreasing on $[0, a]$ hence $G(u(r_2)) > G(\xi)$. On the other hand $G(u(r_2)) \leq G(\xi)$ by (2.1) and this leads to a contradiction. We now show that $u$ remains bounded from above by examining the values of $u \geq z$. In fact, if for some $r_0 > 0$, $u(r_0) = z$ and $u'(r_0) > 0$, then, because $g(s) = 0$ if $s \geq z$ (see Remark 1.3), we have

$$u''(r) + \frac{N-1}{r} u'(r) = 0 \quad \text{for} \quad r \geq r_0,$$

then

$$u'(r) = u'(r_0) \left(\frac{r_0}{r}\right)^{N-1} \leq u'(r_0) =: \beta > 0,$$

thus

$$u(r) \leq z + \beta (r - r_0),$$

hence $u(r)$ remains bounded from above for $r$ bounded. \[Remark 2.2.\] For $\xi \in [0, a]$, we occasionally denote the solution of (1.3) by $u = u(r, \xi)$ if we want to show the dependence on the initial data. In this case $u'$ denotes $u'_r(r, \xi)$.

Let us observe that for $\xi \in [0, a]$, $g(\xi) < 0$ and hence $u''(0, \xi) = -g(\xi) > 0$. This implies that

$$u'(r, \xi) > 0 \quad \text{and} \quad u(r, \xi) < z \quad \text{for small} \quad r > 0.$$ \[2.2\]

Now let

$$S^+ = \{ \xi \in [0, a]; u(r_0, \xi) = z \text{ for some } r_0 = r_0(\xi) > 0 \},$$

$$S^- = \{ \xi \in [0, a]; u'(r_0, \xi) = 0 \text{ for some } r_0 = r_0(\xi) > 0 \text{ and } u(r, \xi) < z \text{ for all } r \geq 0 \},$$

$$S^z = \{ \xi \in [0, a]; u'(r, \xi) > 0 \text{ for all } r > 0 \text{ and } u(r, \xi) < z \text{ for all } r \geq 0 \}.$$
Owing to (2.2), we can easily see that the sets $S^{+}, S^{-}$ and $S^{z}$ are mutually disjoint and cover the interval $[0,\alpha[$. The idea of the proof of the main theorem then goes as follows: we first show that if $\xi \in S^{z}$ then $u = u(r, \xi)$ is the required solution of Theorem 1.1, and next we prove that $S^{z} \neq \emptyset$.

**Lemma 2.3** ($\xi \in S^{z} \Rightarrow \lim_{r \to \infty} u(r, \xi) = z$). Let $\xi \in S^{z}$. Then the number $l = \lim_{r \to \infty} u(r, \xi)$ satisfies $g(l) = 0$, i.e. $l = \alpha$ or $l = z$. Moreover, if $g$ satisfies (1.9), then $l = z$.

**Proof.** Let $\xi \in S^{z}$. The solution $u(r) = u(r, \xi)$, being increasing and bounded from above by $z$, satisfies $u(r) \not\to l \in [\xi, z]$ as $r \to \infty$. The boundedness of $u$ and the regularity of $G$ imply, thanks to (2.1), that $\int_{0}^{\infty} \frac{(u(s))^{2}}{s} \, ds$ is bounded for $r \geq 0$, hence

$$
\int_{0}^{\infty} \frac{(u(s))^{2}}{s} \, ds < \infty.
$$

Then, passing to the limit $r \to \infty$ in (2.1) shows that $u'(r)$ converges as $r \to \infty$ and the limit should be

$$
\lim_{r \to \infty} u'(r) = 0,
$$

since $u$ is bounded. We may now pass to the limit $r \to \infty$ in (1.3) in order to obtain, using the boundedness of $u'$,

$$
\lim_{r \to \infty} u''(r) = 0,
$$

hence

$$
g(l) = 0,
$$

and therefore

$$
l = \alpha \quad \text{or} \quad l = z.
$$

We now assume $g$ satisfies (1.9) and we claim that $l \neq \alpha$.

To understand the idea of the proof, we show the result for sufficiently regular $g$ and $u$, and for (1.9) replaced by $g'(\alpha) > 0$. Differentiating (1.3) with respect to $r$, we get

$$
-u''' = \frac{N - 1}{r} u'' + \frac{N - 1}{r^{2}} u' = g'(u)u'.
$$

Multiplying by $r^{N-1 - \frac{1}{2}}$ and noting that

$$
\left( r^{\frac{N-1}{2}} u' \right)'' = r^{\frac{N-1}{2}} u'' + r^{\frac{N-1}{2}} \left( \frac{N - 1}{r} \right) u'' + r^{\frac{N-1}{2}} \left( \frac{(N - 1)(N - 3)}{4r^2} \right) u'
$$

we obtain

$$
- \left( r^{\frac{N-1}{2}} u' \right)'' = r^{\frac{N-1}{2}} \left( g'(u) - \left( \frac{N - 1}{r^2} + \frac{(N - 1)(N - 3)}{4r^2} \right) \right) u'.
$$

(2.3)

Let

$$
u = r^{\frac{N-1}{2}} u'
$$

then (2.3) becomes

$$
-v'' = \left( g'(u) - \left( \frac{N - 1}{r^2} + \frac{(N - 1)(N + 1)}{4r^2} \right) \right) u',
$$

(2.4)

with $v > 0$ on $]0,\infty[$. We now assume by contradiction that $u(r) \not\to \alpha$ as $r \to \infty$. Then $g'(u(r)) \to g'(\alpha) > 0$ as $r \to \infty$ and, on the other hand, we have $\frac{(N - 1)(N + 1)}{4r^2} \to 0$ as $r \to \infty$. These arguments ensure the existence of $R_{1} > 0$ and $\omega_{1} > 0$ such that

$$
g'(u(r)) - \frac{(N - 1)(N + 1)}{4r^2} \geq \omega_{1} \quad \text{for} \quad r \geq R_{1},
$$
therefore, by (2.4), we finally get
\[-u''(r) \geq \omega_1 v(r) \quad \text{for} \quad r \geq R_1.\] (2.5)

Thus \(u'' < 0\) for \(r \geq R_1\), which implies that \(u'(r) \leq L \in [-\infty, \infty]\) as \(r \to \infty\). If \(L < 0\) then \(u(r) \to -\infty\) as \(r \to \infty\) and this is impossible by the positivity of \(v\). If \(L \geq 0\) then \(u' > 0\) and \(u\) increases on \([R_1, \infty]\), thus \(u(r) \geq v(R_1) > 0\) for \(r \geq R_1\). Again by (2.5) we get \(u''(r) \leq -\omega_1 v(R_1) < 0\) and consequently \(u'(r) \to -\infty\) as \(r \to \infty\) and this is also impossible by the positivity of \(v'\).

This contradiction shows that \(l \neq a\) as required.

We turn back to the general case where \(u\) is only \(C^2\) and \(g\) is a locally Lipschitz function satisfying (1.9). Arguing in a similar manner, we assume \(u(r) \not\to a\) as \(r \to \infty\) and we set
\[v(r) = r^{N-1} (a - u(r)),\]
then \(v > 0\) on \([0, \infty]\). By differentiating the above equation twice in \(r\), and using (1.3), we end up with an equation, similar to (2.4), that reads
\[-u'' = \left(\frac{g(u)}{u - a} - \frac{(N-1)(N-3)}{4r^2}\right)v.\] (2.6)

Since \(u(r) \not\to a\) as \(r \to \infty\), we use condition (1.9) together with the fact that \(\frac{(N-1)(N-3)}{4r^2} \to 0\) as \(r \to \infty\), to deduce the existence of \(R_2 > 0\) and \(\omega_2 > 0\) such that
\[\frac{g(u)}{u - a} - \frac{(N-1)(N-3)}{4r^2} \geq \omega_2 \quad \text{for} \quad r \geq R_2.\]

From (2.6) we are lead to
\[-u''(r) \geq \omega_2 v(r) \quad \text{for} \quad r \geq R_2.\]

Arguing in a similar manner as above, we reach our contradiction and therefore \(l \neq a\).

Lemma 2.3 indicates that it is sufficient to show \(S^z \neq \emptyset\) in order to terminate the proof of Theorem 1.1. Having
\[\{0, a\} = S^+ \cup S^- \cup S^z\] (2.7)
with \(S^+, S^-\) and \(S^z\) mutually disjoint, we can easily see (using the connectedness of \([0, a]\)) that \(S^z \neq \emptyset\) if \(S^z\) are both nonempty open subsets of \([0, a]\). This will be the goal of the following lemmas.

**Lemma 2.4** (\(S^-\) is nonempty). Let \(t_0\) be given by (1.7). Then \([t_0, a] \subset S^-\).

**Proof.** Let \(\xi \in [t_0, a] \subset [0, a] = S^+ \cup S^- \cup S^z\). Assume to the contrary that \(\xi \not\in S^-\), then \(\xi \in S^+ \cup S^z\).

If \(\xi \in S^+\) then there exists \(r_0 > 0\) such that \(u(r_0) = z\). Using (2.1) for \(r = r_0\), we get
\[G(z) + \frac{1}{2} \left(u'(r_0)^2\right)^2 + (N-1) \int_0^{r_0} \frac{(u'(s))^2}{s} ds = G(\xi).\]

Note that \(u'(r_0) \neq 0\) since otherwise \(u \equiv z\) on \([0, \infty]\) by the uniqueness of the solution of the ODE (1.3), which is impossible. This simple observation used in the above equation gives
\[G(z) < G(\xi)\] (2.8)
in contradiction with (1.8).

If \(\xi \in S^z\) then, by Lemma 2.3, we have \(\lim_{r \to \infty} u(r) = z\). Passing to the limit \(r \to \infty\) in (2.1) we obtain
\[G(z) + (N-1) \int_0^{\infty} \frac{(u'(s))^2}{s} ds = G(\xi),\]
where \(\int_0^{\infty} \frac{(u'(s))^2}{s} ds > 0\) as \(u\) increases from \(\xi\) to \(z\) with \(\xi < z\). This leads again to (2.8) and hence a contradiction with (1.8). These contradictions show that \(\xi \in S^-\) as required. \(\square\)
Lemma 2.5 ($S^+$ is nonempty). There exists a sequence $(\xi_n)_{n \geq 1} \subset S^+$ such that $\lim_{n \to \infty} \xi_n = 0$.

The proof of this lemma is based on the following existence result for semilinear elliptic equations on $B = B_R(0) \subset \mathbb{R}^N$ with Dirichlet conditions.

Let $f : [0, \infty) \to \mathbb{R}$ be a locally Lipschitz function with $f(0) = 0$ and $f(\theta) = 0$ for some $\theta > 0$. As before, we may always identify $f$ with its extension

$$
\tilde{f} = \begin{cases} 
  f & \text{on } [0, \theta[, \\
  0 & \text{elsewhere},
\end{cases}
$$

since we are interested in solutions ranging between 0 and $\theta$. We assume the following conditions on $f$ which are similar to (1.5) and (1.6).

There exists $\gamma \in ]0, \theta[ \text{ such that } f(u) < 0 \text{ for } 0 < u < \gamma \text{ and } f(u) > 0 \text{ for } \gamma < u < \theta$. \hspace{1cm} (2.9)

There exists $\zeta > 0$ such that $F(\zeta) > 0$, where $F(t) = \int_0^t f(s) \, ds$. \hspace{1cm} (2.10)

**Theorem 2.6** (Existence of semilinear elliptic equations [1, 9]). Let $f : [0, \infty) \to \mathbb{R}$ be a locally Lipschitz function with $f(0) = 0$ and $f(\theta) = 0$ for some $\theta > 0$. Assume also that $f$ satisfies (2.9) and (2.10). Then, for some $R$ large enough, there exists a positive solution $u$ of (1.2) satisfying $0 \leq u \leq \theta$ and

$$
\int_B F(u) \, dx > 0. \hspace{1cm} (2.11)
$$

Remark 2.7. By the moving plane method [3, 6], the solution $u$ in Theorem 2.6 is radially symmetric whose maximum is attained at 0, i.e. $\max_B u = u(0)$. Moreover, if we define

$$
\zeta_* = \inf \{ \zeta > 0; F(\zeta) > 0 \},
$$

then in view of (2.9) (2.10) and (2.11), $\zeta_* > 0$ exists in $\mathbb{R}$ and

$$
\max_B u = u(0) > \zeta_*.
$$

In fact, if $\max_B u \leq \zeta_*$, then $0 \leq u \leq \zeta_*$, hence $F(u) \leq 0$ on $B$ which violates (2.11).

**Remark 2.8.** If $g$ is the function in Theorem 1.1, then the function $s \mapsto -g(z - s)$ satisfies (2.9) and (2.10) with

$$
\theta = z, \quad \gamma = z - \alpha, \quad \zeta = z - \tau \quad \text{and} \quad \zeta_* = z - \tau_0. \hspace{1cm} (2.12)
$$

We are now ready to prove Lemma 2.5.

**Proof of Lemma 2.5.** Set

$$
f(s) = -g(z - s), \hspace{1cm} (2.13)
$$

then in view of (1.5) and (1.6), it is easy to see that $f$ satisfies (2.9) and (2.10) (see Remark 2.8). For any integer $n \geq 1$, let

$$
\zeta_n = z - \frac{\tau_0}{n}, \hspace{1cm} (2.14)
$$

then

$$
\xi_1 \leq \xi_n < z \quad \text{and} \quad \lim_{n \to \infty} \xi_n = z.
$$
The conditions (2.9), (2.10) satisfied by $f$ together with (2.12), enables us to construct a locally Lipschitz function $f_n : \mathbb{R} \rightarrow \mathbb{R}$ verifying (2.9) with

\[ \theta = z \quad \text{and} \quad \gamma = z - \alpha, \]

and (2.10) with $\zeta, \zeta^*$ may be chosen so that

\[ \zeta_n = \zeta^* < \zeta < z. \]

Moreover,

\[ f_n = f \quad \text{on } ]-\infty, z - \alpha[ \cup ]z, \infty[ \quad \text{and} \quad f_n < f \quad \text{on } ]z - \alpha, z[. \]

Applying Theorem 2.6, there exists, for $R$ sufficiently large, a radially symmetric positive solution $v_n$ of

\[
\begin{cases}
\Delta u + f_n(u) = 0 & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}
\] (2.15)

with

\[ 0 < v_n < z \quad \text{on } B. \]

However, by Remark 2.7, we deduce that

\[ \max_B v_n = v_n(0) > \zeta_n. \] (2.16)

Since

\[ f_n \leq f, \]

and $v_n$ is a solution of (2.15), then $v_n$ is a sub-solution of

\[
\begin{cases}
\Delta u + f(u) = 0 & \text{in } B, \\
u = 0 & \text{on } \partial B.
\end{cases}
\] (2.17)

On the other hand, we have $f(z) = 0$ and consequently the constant function $z$ is a super-solution of (2.17). Therefore, by applying Perron’s method, we infer the existence of a radially symmetric positive solution $w_n = w_n(r = |x|)$ of (2.17) such that

\[ v_n \leq w_n \leq z. \] (2.18)

Thanks to (2.16), (2.18) and the maximum principle, we thus obtain

\[ z > \max_B w_n = w_n(0) \geq v_n(0) > \zeta_n. \] (2.19)

Being a radial solution and by (2.13), the function $w_n$ satisfies

\[
\begin{cases}
-w''_n(r) - \frac{N - 1}{r}w'_n(r) = -g(z - w_n(r)) & \text{for } 0 < r < \infty, \\
w_n(0) > \zeta_n, & w_n(R) = 0 \quad \text{and} \quad w'_n(0) = 0.
\end{cases}
\]

We take

\[ u_n = z - w_n. \]
then \( u_n \) clearly satisfies
\[
\begin{cases}
-u''_n(r) - \frac{N-1}{r} u'_n(r) = g(u_n(r)) & \text{for } 0 < r < \infty, \\
0 < u_n(0) < z - \zeta_n, \quad u_n(R) = z \quad \text{and} \quad u'_n(0) = 0.
\end{cases}
\]

For every integer \( n \geq 1 \), let
\[
\xi_n = u_n(0),
\]
then the previous equation leads
\[
\xi_n \in S^+
\]
and, by (2.14),
\[
\xi_n < \frac{r_0}{n},
\]
hence
\[
\lim_{n \to \infty} \xi_n = 0,
\]
and this terminates the proof.

At last it remains to show that \( S^+ \) and \( S^- \) are both open. This is done by using the continuous dependence of \( u(r, \xi) \) and \( u'(r, \xi) \) on the initial data \( \xi \). More precisely we prove:

**Lemma 2.9.** The sets \( S^\pm \) are open in \( \mathbb{R} \).

**Proof.** We only show the openness of \( S^- \) as the proof that \( S^+ \) is open is quite identical. Let \( \xi \in S^- \), then
\[
r_* = \inf \{ r > 0 ; u'(r, \xi) = 0 \} > 0
\]
and
\[
u'(r, \xi) > 0 \quad \text{for all} \quad r \in ]0, r_*].
\]
From (1.3) we deduce that \( u''(r_*, \xi) = -g(u(r_*, \xi)) \). We claim that \( u''(r_*, \xi) \neq 0 \). If not, then as \( \xi < u(r_*, \xi) < z \), we get
\[
u(r_*, \xi) = a \quad \text{and} \quad u'(r_*, \xi) = 0,
\]
and hence, by the uniqueness of the ODE, we get \( u = a \), which is impossible. Since \( u'(r, \xi) \searrow 0 \) when \( r \searrow r_* \), then \( u''(r_*, \xi) < 0 \) and therefore there exists a number \( r_1 > r_* \) such that
\[
u(r, \xi) < u(r_*, \xi) \quad \text{for all} \quad r \in ]r_*, r_1].
\]
By using a continuity argument, we can easily see that for \( \eta \) near \( \xi \) we have
\[
\begin{cases}
u(r_1, \eta) < u(r_*, \eta), \\
\eta < u(r, \eta) < z \quad \text{for all} \quad r \in ]0, r_1].
\end{cases}
\]
This points out that \( \eta \in S^- \) and so \( S^- \) is open.

**Proof of Theorem 1.1.** Directly follows from (2.7) together with Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.9.
REFERENCES


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