Formal derivation and existence result of an approximate model on dislocation densities

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In this work, we are interested in a mathematical problem arising from the dynamics of dislocation densities in crystals. The model, originally developed by Groma, Czikor, and Zaiser in [8], is a coupled singular parabolic system that describes the motion of dislocations in a bounded crystal, taking into account the short-range interactions and the effect of exterior stresses. We show the derivation and the short time existence and uniqueness of a regular solution in a Hölder space using a fixed point argument and a particular comparison principle.

KEYWORDS
boundary value problems, comparison principle, parabolic systems

AMS CLASSIFICATION
35K50, 35K40, 35K55

1 | INTRODUCTION

Dislocations are line defects, or microscopic irregularity inside materials were the atoms are out of position in the crystalline structure. They represent a non-stationary phenomena when stress fields are applied. The typical thickness of a dislocation line is of order of $10^{-9}$ m and its typical length is of order of $10^{-6}$ m. Mathematically, this theory was originally developed by Volterra [21] in 1907. In 1934, Orowan, [18] Polanyi [19] and Taylor [20] introduced the principal explanation of macroscopic plastic deformation in crystals as a result of the motion of dislocations. The first direct observation of dislocations using electron microscope goes back to Hirsch, Horne, and Whelan [11] and Bollmann [1] in 1956. For a recent study of the physical theory of dislocations, we refer the reader to Nabaro [17] and Hirth and Lothe [10]. However, for a mathematical study of various models on dislocations, we refer to [2–5,16].

In our work, we are interested in the mathematical study of a model on dislocation densities developed by Groma, Czikor, and Zaiser (GCZ for short) in [8] that describes the short-range interactions. This model is written as a coupled system of nonlinear parabolic equations on a bounded domain in one-space dimension. Several variants of the GCZ models have been treated in the work of [6,7,12–14] where particular assumptions on the exterior stress field have been involved. In fact, in their work, they only considered either constant or bounded space-time dependent stresses. This case was far from the real stress in [8] which scales as a nonlinear term (that we precise later) depending on the dislocation density. To our knowledge, our result is the first theoretical one of its type, treating the real physical stress field where we overcome the nonlinear difficulty.

Our main result is to prove the short time existence and uniqueness using a fixed point argument and a comparison principle on the gradient of the solution.

This paper is organized as follows. Section 2 is devoted to the derivation and the general setting of our problem. The principal notations and results are detailed in Section 3. A comparison principle, on the gradient of the solution, is given in Section 4. In Section 5, we use a fixed point argument on convenient spaces to establish the short time existence and uniqueness. We end up in Section 6 with some conclusions.
2 | THE MODELING

2.1 | Physical motivation and setting of the problem

Groma, Czikor, and Zaiser\cite{8} have proposed a model to described the possible accumulation of dislocations on the boundary layer of a material. They have presented a two-dimensional model describing the dynamics of parallel edge dislocations densities in a bounded three-dimensional crystal. To more explain the physical model, we can take a channel (see Figure 1 below) that contains a certain number of parallel edge dislocations and bounded by walls that are impenetrable by dislocations. This channel has a finite width in the $x$-direction and an infinite extension in the $y$-direction. The dislocation lines $\xi$ are supposed to be perpendicular to $x$ and $y$. The Burgers vector $\vec{b}$ is parallel to $x$-direction. The direction of Burgers vector determine the classification (positive or negative) of edge dislocations.

On our work, we derive our mathematical model on dislocation dynamics with short-range interaction. All our computations are inspired by GCZ\cite{8} where we suppose a particular configuration of the dislocation lines.

Let $\theta_+$ and $\theta_-$ be the densities of the positive and negative dislocations, respectively. Denote by

$$\phi = \theta_+ + \theta_-$$

the total dislocation density and by

$$\psi = \theta_+ - \theta_-$$

the sign dislocation density. In the case of no short range dislocation-dislocation interactions, the model satisfied by $\phi$ and $\psi$ is the following:

$$\begin{align*}
\frac{\partial \phi}{\partial t}(x,t) + \frac{\partial}{\partial x} \left[ \psi(x,t) \{ \tau_{sc}(x,t) + \tau_{ext} \} \right] &= 0, \\
\frac{\partial \psi}{\partial t}(x,t) + \frac{\partial}{\partial x} \left[ \phi(x,t) \{ \tau_{sc}(x,t) + \tau_{ext} \} \right] &= 0
\end{align*}$$

(2.1)

where $\tau_{ext}$ is the external stress, and $\tau_{sc}$ is the self-consistent stress given by

$$\tau_{sc}(x,t) = \int_0^1 \psi(y,t) \tau_{ind}(x-y,t) dy,$$  

(2.2)

where $\tau_{ind}(x_i-x_j)$ is the shear stress created at $x_i$ by a positive dislocation located at $x_j$; $\tau_{ind}(x)$ is proportional to $\frac{1}{x}$. 

![Figure 1](image)
In the case of short range dislocation-dislocation interactions, the model satisfied by $\phi$ and $\psi$ can be written as:

\[
\begin{align*}
\frac{\partial \phi}{\partial t}(x,t) + \frac{\partial}{\partial x} \left[ \psi(x,t) \left( \tau_{sc}(x,t) + \tau_{ext} - \tau_f + \tau_b \right) \right] &= 0, \\
\frac{\partial \psi}{\partial t}(x,t) + \frac{\partial}{\partial x} \left[ \phi(x,t) \left( \tau_{sc}(x,t) + \tau_{ext} - \tau_f + \tau_b \right) \right] &= 0
\end{align*}
\]  

(2.3)

where $\tau_f$ is the local flow stress given by

\[
\tau_f = \frac{AC}{2} b \sqrt{\phi},
\]

(2.4)

with $A = \mu /[2\pi(1-\nu)]$, $\mu$ is the shear modulus, $\nu$ is the Poisson ratio, $b$ is the modulus of the Burgers vector, and $C = \int_0^1 \int_0^1 \frac{x^2(y^2-x^2)}{(x^2+y^2)^2} p(x,y) dxdy$. The stress contribution $\tau_f$ is related to the correlation function $p(x,y)$ which in physical terms characterizes the polarization of dipoles of dislocations of opposite signs. Moreover, $\tau_b$ is the back stress and is given by

\[
\tau_b = -\frac{AD\bar{b}}{\phi} \nabla \psi,
\]

(2.5)

where $D = \int_0^1 \int_0^1 \frac{x^2(y^2-x^2)}{(x^2+y^2)^2} p(x,y) dxdy$ is a non-dimensional constant, and $p(x,y)$ is a given function. Let

\[
\tau_{eff} = \tau_{ext} - \tau_f.
\]

In a special physical setting depending on the internal material structure and the frame of reference, we may assume $\bar{b} = (1,0)$ and $A = D = 1$. This will also simplify the presentation of the mathematical model. Plug (2.5) into (2.3), we obtain, after setting all constants to be 1:

\[
\begin{align*}
\frac{\partial \phi}{\partial t}(x,t) + \frac{\partial}{\partial x} \left[ \psi(x,t) \left( \tau_{sc}(x,t) + \tau_{eff} - \frac{1}{\phi} \frac{\partial \psi}{\partial x} \right) \right] &= 0, \\
\frac{\partial \psi}{\partial t}(x,t) + \frac{\partial}{\partial x} \left[ \phi(x,t) \left( \tau_{sc}(x,t) + \tau_{eff} - \frac{1}{\phi} \frac{\partial \psi}{\partial x} \right) \right] &= 0
\end{align*}
\]  

(2.6)

in the case of a constrained channel deforming in simple shear, i.e. a channel of width $L$ in the $x$ direction and infinite extension in the $y$ direction, bounded by walls that are impenetrable by dislocations (the plastic deformation at the walls is zero). The slip direction corresponds to the $x$ direction, and the layer is sheared by a constant shear stress.

### 2.2 A simplified model

The system (2.6) is a simplified version of a system studied by Van der Giessen and co-workers using both discrete dislocation and continuum plasticity approaches. The simplifications stem out from the fact that only a single slip system is assumed to be active, such that reactions between dislocations need not to be considered, and that the boundary conditions reduce to “no flux” conditions for the dislocation fluxes at the boundary walls.

The system envisaged is particularly simple also because it is homogeneous in the $y$ direction. In other words, the problem is assumed to be invariant by translation in the $y$ direction. In order to better explain this point, we suppose that, if $(S)$ is a cross sectional surface perpendicular to the dislocation lines, then the arrangement of dislocation points in $(S)$ is invariant by translation in the $y$ direction. In this case, we deduce that studying the dynamics of dislocation points on the line $(\mathcal{L})$ (see Figure 2) gives the complete information of the dynamics of dislocation lines in the channel. As a summary, we can write that

\[
\text{dynamics in } (\mathcal{L}) \Rightarrow \text{dynamics in } (S) \Rightarrow \text{dynamics in the bounded channel.}
\]

It follows from equation 2.2 that in this case the long-range self-consistent stress field is zero for an arbitrary function $\kappa(x)$

\[
\tau_{sc} = 0,
\]
i.e. any dislocation interaction in the system is of short-range nature and hence it is described by the flow stress $\tau_f$ and the gradient dependent stress $\tau_b$. Assuming moreover that $\tau_{ext} = 0$, system (2.6) becomes

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x} \left[ \psi \left\{ -\tau_f - \frac{1}{\phi} \frac{\partial \psi}{\partial x} \right\} \right] &= 0, \\
\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x} \left[ \phi \left\{ -\tau_f - \frac{1}{\phi} \frac{\partial \psi}{\partial x} \right\} \right] &= 0.
\end{aligned}
$$

(2.7)

Since $\tau_f$ scales as $\sqrt{\phi}$, (2.7) can be rewritten:

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x} \left[ \psi \left\{ -\sqrt{\phi} - \frac{1}{\phi} \frac{\partial \psi}{\partial x} \right\} \right] &= 0, \\
\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x} \left[ \phi \left\{ -\sqrt{\phi} - \frac{1}{\phi} \frac{\partial \psi}{\partial x} \right\} \right] &= 0.
\end{aligned}
$$

(2.8)

Writing (2.8) in terms of $\theta^\pm$, we obtain:

$$
\begin{aligned}
\theta_i^+ &= \left( \sqrt{\theta^+ + \theta^-} + \frac{\theta^+ x - \theta^- x}{\theta^+ + \theta^-} \right) \theta^+, \\
\theta_i^- &= \left( -\sqrt{\theta^+ + \theta^-} + \frac{\theta^+ x - \theta^- x}{\theta^+ + \theta^-} \right) \theta^-.
\end{aligned}
$$

(2.9)

We now pose

$$
\rho^\pm_x = \theta^\pm, \quad \rho = \rho^+ - \rho^-, \quad \kappa = \rho^+ + \rho^-,
$$

where $\rho^+, \rho^-$ are the unknown scalars, that we denote for simplicity by $\rho^\pm$. Here, the difference $\rho^+ - \rho^-$ represents the plastic deformation.

Subtract the two equations of (2.9) and integrate with respect to the space variable, we obtain

$$
\theta_i^+ - \theta_i^- = \left[ \left( \sqrt{\theta^+ + \theta^-} + \frac{\theta^+ x - \theta^- x}{\theta^+ + \theta^-} \right) \theta^+ \right]_x
$$

hence

$$
\int_0^1 (\theta_i^+ - \theta_i^-) \, dx = \left( \sqrt{\theta^+ + \theta^-} + \frac{\theta^+ x - \theta^- x}{\theta^+ + \theta^-} \right) (\theta^+ + \theta^-),
$$
thus
\[\frac{\partial}{\partial t} \int_{0}^{1} (\theta^+ - \theta^-) \, dx = \left( \sqrt{\theta^+ + \theta^-} + \frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} \right) (\theta^+ + \theta^-),\]
therefore
\[\frac{\partial}{\partial t} \int_{0}^{1} (\rho_x^+ - \rho_x^-) \, dx = \left( \sqrt{\rho_x^+ + \rho_x^-} + \frac{\rho_{xx}^+ - \rho_{xx}^-}{\rho_x^+ + \rho_x^-} \right) (\rho_x^+ + \rho_x^-),\]
then
\[\frac{\partial}{\partial t} \int_{0}^{1} \rho_x \, dx = \left( \sqrt{k_x^+ + \frac{\rho_{xx}}{k_x^+}} \right) \rho_x,\]
thus we conclude
\[\rho_t = k_x \sqrt{k_x^+ + \rho_{xx}}.\]

Now let us add the two equations of (2.9) and integrate with respect to the space variable, we obtain
\[\theta_t^+ + \theta_t^- = \left[ \left( \sqrt{\theta^+ + \theta^-} + \frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} \right) (\theta^+ + \theta^-) \right],\]
hence
\[\int_{0}^{1} (\theta_t^+ + \theta_t^-) \, dx = \left( \sqrt{\theta^+ + \theta^-} + \frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} \right) (\theta^+ - \theta^-),\]
thus we get
\[\frac{\partial}{\partial t} \int_{0}^{1} (\theta_t^+ + \theta_t^-) \, dx = \left( \sqrt{\theta^+ + \theta^-} + \frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} \right) (\theta^+ - \theta^-),\]
therefore
\[\frac{\partial}{\partial t} \int_{0}^{1} (\rho_x^+ + \rho_x^-) \, dx = \left( \sqrt{\rho_x^+ + \rho_x^-} + \frac{\rho_{xx}^+ - \rho_{xx}^-}{\rho_x^+ + \rho_x^-} \right) (\rho_x^+ + \rho_x^-),\]
then
\[\frac{\partial}{\partial t} \int_{0}^{1} \kappa_x \, dx = \left( \sqrt{k_x^+ + \frac{\rho_{xx}}{k_x^+}} \right) \rho_x,\]
thus we have
\[\kappa_t = \rho_x \sqrt{k_x^+ + \frac{\rho_x \rho_{xx}}{k_x^+}},\]
and we conclude that
\[\kappa_t \kappa_x = \rho_t \rho_x.\]

System (2.9) now becomes
\[
\begin{aligned}
\kappa_t \kappa_x &= \rho_t \rho_x, \\
\rho_t &= \rho_{xx} + k_x \sqrt{k_x^+},
\end{aligned}
\tag{2.10}
\]

Remark that, in order to do the above calculations, \(\theta^+\) and \(\theta^-\) are assumed to be sufficiently regular, say \(C^{2+\alpha, 2+\alpha}_x\), where \(C^{1,1/2}\) is the Hölder space that will be defined in Section 3.
We first present result of short time existence and uniqueness of solutions to problem (2.11)–(2.13).

Regularizing equation 2.9 by adding $\epsilon \theta_{z,x}$ and writing down the equations in $\rho$ and $\kappa$, we get (for $T > 0$):

\[
\begin{align*}
\kappa_t &= \epsilon \kappa_{xx} + \frac{\rho_x \rho_{xx}}{\kappa_x} + \frac{\kappa_x}{\sqrt{\kappa_x}} & \text{on } I_T, \\
\rho_t &= (1 + \epsilon) \rho_{xx} + \kappa_x \sqrt{\kappa_x} & \text{on } I_T.
\end{align*}
\]  

The initial data are:

\[
\rho(x, 0) = \rho^0(x) \quad \text{and} \quad \kappa(x, 0) = \kappa^0(x) \quad x \in I,
\]  

and the boundary data are of Dirichlet type:

\[
\rho(0, t) = \rho(1, t) = \kappa(0, t) = 0 \quad \text{and} \quad \kappa(1, t) = 1 \quad t > 0.
\]

3 | NOTATIONS AND THE MAIN RESULTS

3.1 | Notation

Let $T > 0$ and $I_T$ is the cylinder $I \times (0, T)$ with $I = (0, 1)$; $\overline{I}$ is the closure of $I$; $\overline{I}_T$ is the closure of $I_T$; $\partial I$ is the boundary of $I$. The indicator function $1_A$ of the set $A \subseteq \mathbb{R}$ is used when convenient. The letter $c$ will denote constant taking different values, to be determinate in each case. We denote by $D_z^k(u) = \frac{\partial^k u}{\partial x^k}$, where $u$ is a function depending on the parameter $z$ and $k \in \mathbb{N}$. We will use the following spaces:

- The Lesbegue spaces $L^p(I_T)$, $1 \leq p \leq \infty$ with norms $\|u\|_{L^p(I_T)}$;
- The parabolic Sobolev space $Y$, defined by: $Y = W^{2,1}_p(I_T) = \{u \in L^p(I_T) ; D_t^r D_x^s u \in L^p(I_T) \}$ for $2r + s \leq 2$, equipped with the norm $\|u\|_Y = \sum_{2r+s \leq 2} \|D_t^r D_x^s u\|_{L^p(I_T)}$;
- The Hölder space $C^{l/2}(\overline{I}_T)$, $l > 0$ a non-integral positive number, is the Banach space of functions $v(x, t)$ that are continuous in $\overline{I}_T$, together with all derivatives of the form $D_t^r D_x^s v$ for $2r + s < l$, and have a finite norm $\|v\|_{I_T}^{(l)} = \langle v \rangle_{I_T}^{(l)} + \sum_{j=0}^{(l)} \langle v \rangle_{I_T}^{(j)}$, where

\[
\langle v \rangle_{I_T}^{(0)} = \|v\|_{\infty,I_T}, \quad \langle v \rangle_{I_T}^{(j)} = \sum_{2r+s=j} \|D_t^r D_x^s v\|_{I_T}^{(0)}, \quad \langle v \rangle_{I_T}^{(l)} = \langle v \rangle_{I_T}^{(l)} + \sum_{j=0}^{(l)} \langle v \rangle_{I_T}^{(j)}.
\]

and

\[
\langle v \rangle_{x,I_T}^{(l)} = \sum_{2r+s=l} \langle D_t^r D_x^s v \rangle_{x,I_T}^{(l-2)}, \quad \langle v \rangle_{t,I_T}^{(l)} = \sum_{0<s<l/2} \langle D_t^r D_x^s v \rangle_{t,I_T}^{(l/2)},
\]

with

\[
\langle v \rangle_{x,I_T}^{(a)} = \inf \{c; |v(x, t) - v(x', t)| \leq c|x - x'|^a, (x, t), (x', t) \in \overline{I}_T\}, \quad 0 < a < 1,
\]

\[
\langle v \rangle_{t,I_T}^{(a)} = \inf \{c; |v(x, t) - v(x, t')| \leq c|t - t'|^a, (x, t), (x, t') \in \overline{I}_T\}, \quad 0 < a < 1.
\]

3.2 | The Main result

We first present result of short time existence and uniqueness of solutions to problem (2.11)–(2.13).
Theorem 3.1 (Short time existence and uniqueness). Let $\epsilon > 0$ a fixed constant. Let $\rho^0$ and $\kappa^0 \in C^\infty(\bar{T})$, such that $\rho^0(0) = \rho^0(1) = 0$ and $\kappa^0(0) = 0$, $\kappa^0(1) = 1$ with

$$k^0_x > |\rho^0_x| \quad \text{on} \quad \bar{T}. \tag{3.14}$$

Assume

$$\begin{cases}
(1 + \epsilon)\rho^0_{xx} + \kappa^0_x \sqrt{\kappa^0_x} = 0 & \text{on} \quad \partial I, \\
(1 + \epsilon)\kappa^0_{xx} + \rho^0_x \sqrt{\kappa^0_x} = 0 & \text{on} \quad \partial I. \tag{3.15}
\end{cases}$$

Then there exists a short time $T > 0$ and a unique solution $(\rho, \kappa)$ of problem (2.11)–(2.13), satisfying

$$(\rho, \kappa) \in \left( C^{3+\alpha}, \frac{3+\alpha}{2} (\bar{I}_T) \cap C^\infty(\bar{I} \times (0, T)) \right)^2,$$

with $0 < \alpha < 1$ and

$$\kappa_x > |\rho_x| \quad \text{on} \quad \bar{I} \times (0, T). \tag{3.16}$$

4 | A COMPARISON PRINCIPLE

In this section, and for symmetry reasons, we take $I = (-1, 1)$ and

$$G_a(x) := \sqrt{x^2 + a^2}, \quad x, a \in \mathbb{R}.$$

Some basic properties on the positive function $G_a$ will be frequently utilized. Mainly:

$$\begin{cases}
G_a(x)G'_a(x) = x, \\
G^3_a(x)G''_a(x) = a^2.
\end{cases}$$

Proposition 4.1. Let $(\rho, \kappa)$ be a regular solution of (2.11) on the compact $\bar{I}_T$ with $\kappa_x > 0$, and the initial data $(\rho^0, \kappa^0)$ satisfies:

$$\kappa^0_x \geq G_{\gamma_0}(\rho^0_x), \quad \gamma_0 \in (0, 1). \tag{4.17}$$

Then there exists a $C^1$ positive function $\gamma : [0, T) \to ]0, \infty[\text{ such that}$

$$\kappa_x \geq G_{\gamma}(\rho_x) \quad \text{on} \quad I_T. \tag{4.18}$$

In all what follows, we use the following notation:

$$M := \kappa_x - G_{\gamma}(\rho_x). \tag{4.19}$$

where the function $\gamma$ is to be determined in a way that $\kappa_x \geq G_{\gamma}(\rho_x)$ on $I_T$ (i.e. $M \geq 0$).

To prove Proposition 4.1, we shall first prove two lemmas.

Lemma 4.2. There exists a $C^1$ positive function $\gamma : [0, T) \to ]0, \infty[\text{ such that } M$, defined in (4.19), satisfies:

$$M_t \geq \epsilon M_{xx} + AM_x + BM, \tag{4.20}$$

where $A$ and $B$ are two functions given by (4.25) and (4.30), respectively.
Proof. Assuming the regularity of \( \rho, \kappa \) and \( \gamma \), we compute:

\[
\begin{align*}
M_t &= \kappa_{tt} - G'_\gamma(\rho_x)\rho_{xt} - \Gamma, \\
M_x &= \kappa_{xx} - G'_\gamma(\rho_x)\rho_{xx}, \\
M_{xx} &= \kappa_{xxx} - G''_\gamma(\rho_x)\rho_{xx}^2 - G'_\gamma(\rho_x)\rho_{xxx},
\end{align*}
\]  

(4.21)

where

\[
\Gamma = \frac{\gamma'\gamma''}{\sqrt{\rho_x^2 + \gamma^2}}.
\]  

(4.22)

Differentiating (2.11) with respect to \( x \), we easily obtain:

\[
\begin{align*}
\kappa_{xt} &= \varepsilon \kappa_{xxx} + \frac{\rho_x^2}{\kappa_x^2} + \rho_x \rho_{xxx} - \frac{\rho_x \rho_{x\kappa x}}{\kappa_x^2} + \rho_{xx} \sqrt{\kappa_x} + \frac{\rho_x \kappa_{xx}}{2 \sqrt{\kappa_x}}, \\
\rho_{xt} &= (1 + \varepsilon) \rho_{xxx} + \frac{3}{2} \kappa_{xx} \sqrt{\kappa_x}.
\end{align*}
\]  

(4.23)

Using (4.21) and (4.23), direct and initial computations lead:

\[
M_t = \varepsilon M_{xx} + A M_{xx} + B_1 M + L_1 + L_2 - \Gamma,
\]  

(4.24)

with

\[
\begin{align*}
A &= -\frac{\rho_x \rho_{xx}}{\kappa_x^2} + \frac{\rho_x}{2 \sqrt{\kappa_x}} - \frac{3}{2} G'_\gamma(\rho_x) \sqrt{\kappa_x}, \\
B_1 &= \frac{\rho_x^2}{\kappa_x^2} + \frac{\rho_x}{\sqrt{\kappa_x}} - \frac{G'_\gamma(\rho_x) \rho_{xxx}}{\kappa_x}, \\
L_1 &= \frac{G'_\gamma(\rho_x) \rho_{xx}^2}{\kappa_x^2} - \frac{G'_\gamma(\rho_x) \rho_{xx} \rho_{xx}^2}{\kappa_x^2}, \\
L_2 &= \varepsilon G''_\gamma(\rho_x) \rho_{xx}^2 + L_3, \\
L_3 &= \frac{G'_\gamma(\rho_x) \rho_{xx} \rho_{xx}^2}{2 \sqrt{\kappa_x}} + \frac{G'_\gamma(\rho_x) \rho_{xx}}{\sqrt{\kappa_x}} - \frac{3}{2} (G'_\gamma(\rho_x))^2 \rho_{xx} \sqrt{\kappa_x}.
\end{align*}
\]  

(4.25)

By estimating the term \( L_1 \) in (4.25), we write:

\[
L_1 = \rho_{xx}^2 (G'_\gamma(\rho_x) - \rho_x G''_\gamma(\rho_x))
\]

\[
= \frac{\rho_{xx}^2}{\kappa_x^2} \frac{\gamma^2}{\sqrt{\rho_x^2 + \gamma^2}} \geq 0.
\]  

(4.26)

The estimate of \( L_2 \) requires to estimate \( L_3 \) first. In fact we have:

\[
L_3 = \frac{\rho_{xx}^2}{2 \sqrt{\kappa_x}} \left( G'_\gamma(\rho_x) \rho_x + 2 G'_\gamma(\rho_x) - 3 (G'_\gamma(\rho_x))^2 \kappa_x \right).
\]
A simple computation shows that:

\[ G'_\nu(\rho_x) \rho_x + 2G_\nu(\rho_x) = \frac{3\rho_x^2 + 2\gamma^2}{G(\rho_x)}, \]

therefore, by elementary properties of the function \( G_\nu \), we get:

\[
L_3 = \frac{\rho_{xx}}{2\sqrt{\kappa_x}} \left( \frac{3\rho_x^2 + 2\gamma^2}{G(\rho_x)} - 3(G'_\nu(\rho_x))^2\kappa_x \right)
= \frac{\rho_{xx}}{2\sqrt{\kappa_x}} \left( 3\rho_x \left( \rho_x - G'_\nu(\rho_x)\kappa_x \right) + 2\gamma^2 \right)
= \frac{\rho_{xx}}{2\sqrt{\kappa_x}} \left( 3\rho_x^2 (G_\nu(\rho_x) - \kappa_x) + 2\gamma^2 G_\nu(\rho_x) \right).
\]

By using the definition of \( M \), we finally arrive to:

\[
L_3 = - \left( \frac{3\rho_x^2 \rho_{xx}}{2G'_\nu(\rho_x)\sqrt{\kappa_x}} \right) M + \frac{\gamma^2 \rho_{xx}}{G_\nu(\rho_x)\sqrt{\kappa_x}},
\]

hence

\[
L_2 = \varepsilon G''_\nu(\rho_x) \rho_{xx}^2 - \left( \frac{3\rho_x^2 \rho_{xx}}{2G'_\nu(\rho_x)\sqrt{\kappa_x}} \right) M + \frac{\gamma^2 \rho_{xx}}{G_\nu(\rho_x)\sqrt{\kappa_x}}.
\] (4.27)

Using (4.24), (4.25), (4.26), and (4.27), we finally deduce that:

\[
M_t \geq \varepsilon M_{xx} + AM_x + B_2 M + \varepsilon G''_\nu(\rho_x) \rho_{xx}^2 + \frac{\gamma^2 \rho_{xx}}{G_\nu(\rho_x)\sqrt{\kappa_x}} - \Gamma,
\] (4.28)

where

\[
B_2 = B_1 - \frac{3\rho_x^2 \rho_{xx}}{2G'_\nu(\rho_x)\sqrt{\kappa_x}}.
\]

To study the term \( \varepsilon G''_\nu(\rho_x) \rho_{xx}^2 \) in (4.28), we can easily write:

\[
\varepsilon G''_\nu(\rho_x) \rho_{xx}^2 = \frac{\varepsilon \kappa_x G''_\nu(\rho_x) \rho_{xx}^2}{\kappa_x} = \frac{\varepsilon (M + G_\nu(\rho_x)) G''_\nu(\rho_x) \rho_{xx}^2}{\kappa_x} = \left( \frac{\varepsilon G''_\nu(\rho_x) \rho_{xx}^2}{\kappa_x} \right) M + \frac{\varepsilon \gamma^2 \rho_{xx}}{(\rho_x^2 + \gamma^2)\kappa_x}.
\]

Therefore (4.28) gives:

\[
M_t \geq \varepsilon M_{xx} + AM_x + BM + \frac{\varepsilon \gamma^2 \rho_{xx}}{(\rho_x^2 + \gamma^2)\kappa_x} + \frac{\gamma^2 \rho_{xx}}{G_\nu(\rho_x)\sqrt{\kappa_x}} - \Gamma,
\] (4.29)

where

\[
B = B_2 + \frac{\varepsilon G''_\nu(\rho_x) \rho_{xx}^2}{\kappa_x}.
\] (4.30)

Finally, we use \( 0 < \gamma < 1 \) to estimate the term \( \frac{\gamma^2 \rho_{xx}}{G_\nu(\rho_x)\sqrt{\kappa_x}} \) in (4.29). Then

\[
G_\nu(\rho_x(x,t)) = \sqrt{\rho_x^2(x,t) + \gamma^2(t)} \geq \alpha(t)(\rho_x^2(x,t) + \gamma^2(t))^{3/4},
\]
where
\[ \alpha(t) = \frac{1}{\left( \| \rho_x(\cdot, t) \|_{L^\infty(I)}^2 + 1 \right)^{1/4}}. \]

We now compute:
\[ \gamma^2 |\rho_{xx}| \leq \frac{\gamma^2 |\rho_{xx}|}{G_{\gamma}(\rho_x) \sqrt{\kappa_x}} \leq \frac{\gamma |\rho_{xx}|}{\alpha(t)(\rho_x^2 + \gamma^2)^{1/2} \sqrt{\kappa_x}} \cdot \frac{\gamma}{\alpha(t)(\rho_x^2 + \gamma^2)^{1/4}}, \]

therefore, by Young's inequality, we get
\[ \gamma^2 |\rho_{xx}| \leq \frac{\varepsilon \gamma^2 \rho_{xx}^2}{(\rho_x^2 + \gamma^2)\kappa_x} + \frac{\gamma^2}{4\varepsilon \alpha^2(t)(\rho_x^2 + \gamma^2)^{1/2}}. \] (4.31)

We want to choose \( \gamma \) such that the differential inequality in \( M \) is homogeneous. In fact, using (4.29) and (4.31), we can infer that:
\[ M_t \geq \varepsilon M_{xx} + AM_x + BM - \frac{\gamma^2}{4\varepsilon \alpha^2(t)(\rho_x^2 + \gamma^2)^{1/2}} - \Gamma. \] (4.32)

It is worth noticing that the term \( \frac{\varepsilon \gamma^2 \rho_{xx}^2}{(\rho_x^2 + \gamma^2)\kappa_x} \) in (4.29) has been eliminated by the special application (4.31).
At this stage, choosing
\[ \gamma(t) = \gamma_0 e^{-\int_0^t \frac{dt}{4\varepsilon \alpha^2(s)}} \] (4.33)
we finally deduce (see (4.32) and the definition (4.22) of \( \Gamma \)) that:
\[ M_t \geq \varepsilon M_{xx} + AM_x + BM, \] (4.34)
and, on the other side, we can easily justify our claim \( 0 < \gamma < 1 \) as \( \gamma_0 \in (0, 1) \).

**Lemma 4.3.** Let \( \theta \in \mathbb{R} \) and \( \overline{M}(x, t) = \cosh(\theta x) M(x, t) \). Under the condition of Lemma 4.2, the function \( \overline{M} \) can not have a positive minimum on \( \partial I \).

**Proof.** Since \( \rho \) and \( \kappa \) are constants on the boundary \( \partial I \times [0, T] \), we obtain:
\[
\begin{aligned}
\varepsilon \kappa_{xx} + \frac{\rho_x \rho_{xx}}{\kappa_x} + \rho_x \sqrt{\kappa_x} &= 0 \quad \text{on } \partial I \times [0, T] \\
(1 + \varepsilon) \rho_{xx} + \kappa_x \sqrt{\kappa_x} &= 0 \quad \text{on } \partial I \times [0, T].
\end{aligned}
\]

therefore
\[ M_x = \frac{\sqrt{\kappa_x}}{1 + \varepsilon} G_{\gamma}(\rho_x) M \quad \text{on } \partial I \times [0, T]. \] (4.35)

Indeed,
\[
M_x = \kappa_{xx} - G_{\gamma}'(\rho_x) \rho_{xx} = -\frac{\rho_x \sqrt{\kappa_x}}{1 + \varepsilon} + G_{\gamma}'(\rho_x) \frac{\kappa_x \sqrt{\kappa_x}}{1 + \varepsilon} = \frac{\sqrt{\kappa_x}}{1 + \varepsilon} G_{\gamma}'(\rho_x) \left( \frac{-\rho_x}{G_{\gamma}'(\rho_x)} + \kappa_x \right).
\]
\[
\begin{align*}
G'_r(\rho_x)(-G'_r(\rho_x) + \kappa_x) \\
&= \frac{\sqrt{\kappa_x}}{1 + \varepsilon} G'_r(\rho_x)M.
\end{align*}
\]

Here we would like to show that the minimum of \( M(\cdot, t) \) is not attained on the boundary and then to use (4.29) to show its positivity. However, the above boundary equation 4.35 carries no information that violates the presence of the minimum on \( \partial I \).

For this reason, we carefully multiply \( M \) by a suitable positive function having large values on the boundary. In particular, we consider \( \tilde{M} \) defined by:

\[
\tilde{M}(x, t) = \cosh(\theta x)M(x, t), \quad \theta \in \mathbb{R}.
\]

Elementary computations show:

\[
\tilde{M}_x = \Theta \tilde{M} \quad \text{on} \quad \partial I \times [0, T], \quad (4.36)
\]

with

\[
\Theta = \left( \theta \tanh(\theta x) + \frac{\sqrt{\kappa_x}}{1 + \varepsilon} G'_r(\rho_x) \right).
\]

The boundedness of \( \frac{\sqrt{\kappa_x}}{1 + \varepsilon} G'_r(\rho_x) \) on \( I_T \) permits the existence of \( \theta \) large enough so that \( \Theta(1, t) > 0 \) and \( \Theta(-1, t) < 0 \) for all \( t \in [0, T] \). Hence, by (4.36), the function \( \tilde{M}(\cdot, t) \) cannot have a positive minimum on \( \partial I \).

We can now present the proof of Proposition 4.1.

\textbf{Proof of Proposition 4.1.} First, we write the partial differential inequality satisfied by \( \tilde{M} \):

\[
\tilde{M}_t \geq \varepsilon \tilde{M}_{xx} + \left( \frac{\rho_x^2}{\kappa_x^2} + \frac{\rho_x}{2\sqrt{\kappa_x}} - \frac{3}{2} \frac{G'_r(\rho_x)\sqrt{\kappa_x}}{\kappa_x} - 2\theta \varepsilon \tanh(\theta x) \right) \tilde{M}_x
\]

\[
+ \left[ \frac{\rho_{xx}^2}{\kappa_x^2} + \frac{\rho_{xx}}{2\sqrt{\kappa_x}} - \frac{3}{2} \frac{G'_r(\rho_x)\rho_{xxx}}{\kappa_x} + \frac{\varepsilon G''_r(\rho_x)\rho_{xx}^2}{2\kappa_x} \right] \tilde{M}.
\]

(4.37)

Due to the regularity of \( \tilde{M} \), we may find a curve \( t \mapsto (x_0(t), t) \) such that

\[
\min_{x \in I} \tilde{M}(x, t) = \tilde{M}(x_0(t), t) = \tilde{m}(t) \quad t \in [0, T].
\]

Without loss of generality, we may always assume \( \kappa_x^0 > \sqrt{(\rho_x^0)^2 + \gamma_0^2} \). In fact, it suffices to adjust \( \gamma_0 \) in (4.17). Therefore (see Step 2):

\[
\tilde{m}(0) > 0 \quad \text{and} \quad x_0(0) \in I.
\]

Again, the regularity of \( \tilde{M} \) ensures

\[
x_0(t) \in I \quad \text{for all} \quad 0 \leq t \leq t_0 \leq T.
\]

We claim that \( t_0 = T \). Indeed, if not, we get \( x_0(t_0) \in \partial I \). Let us show that this can not be true. Indeed, since \( x_0(t) \in I \) for \( 0 \leq t < t_0 \) we directly obtain:

\[
\tilde{M}_x(x_0(t), t) = 0 \quad \text{and} \quad \tilde{M}_{xx}(x_0(t), t) \geq 0.
\]
Moreover, using (4.37), we obtain, for some constant $c \in \mathbb{R}$, the following ordinary differential inequality involving $\overline{m}$:

$$\overline{m}' \geq c\overline{m} \quad \text{for} \quad 0 \leq t < t_0,$$

and therefore, using Grönwall’s inequality, we get

$$\overline{m}(t) \geq \overline{m}(0)e^{ct} \quad \text{for} \quad 0 \leq t < t_0. \quad (4.38)$$

Since $\overline{m}(0) > 0$, the above inequality gives $\overline{m}(t_0) > 0$. Consequently (see Step 2) $x_0(t_0) \in I$ and the claim holds true.

Now as we get $x_0(t) \in I$ for all $t \in [0, T]$, the inequality (4.38) also holds true for all $t \in [0, T]$ with $\overline{m}(0) > 0$. Hence, we can infer that $M \geq 0$ on $I_T$ and therefore $\overline{M} \geq 0$ on $I_T$.

\[ \square \]

5 | SHORT TIME EXISTENCE AND UNIQUENESS

In this section we prove the short time existence and uniqueness for (2.11), (2.12), and (2.13). The main idea is to find a solution of a truncated system where we carefully truncate the gradients $\rho_x$ and $\kappa_x$. To be more precise, we consider, for real values $0 \leq a \leq b$, the two real valued functions $I_a$ and $J_{a,b}$ defined by:

$$I_a(x) = x \mathbb{1}_{|x| \leq a} + a \mathbb{1}_{x \geq a} - a \mathbb{1}_{x \leq -a},$$

and

$$J_{a,b}(x) = \sqrt{x} \mathbb{1}_{a \leq x \leq b} + \sqrt{b} \mathbb{1}_{x \geq b} + \sqrt{a} \mathbb{1}_{x \leq a}.$$

It is easily seen that $I_a$ is a truncation of the identity function, while $J_{a,b}$ is a truncation of $\sqrt{x}$. Remark that both functions are bounded Lipschitz continuous on $\mathbb{R}$, and this property will be repeatedly used hereafter in this section. The value $p > 3$ is taken to emphasis some regularity properties on the solution. Indeed, it is well known (see [15] for the details) that if $p > 3$ then $Y$ is continuously embedded in a parabolic Hölder space:

$$Y \hookrightarrow C^{1+\alpha, \frac{3}{2}}(I_T), \quad \alpha = 1 - \frac{3}{p}, \quad (5.39)$$

with the following fundamental estimate, valid for $u = 0$ on the parabolic boundary

$$\partial^p(I_T) = (\partial I \times [0, T]) \cup (I \times \{0\})$$

that reads:

$$\|u_x\|_{L^\infty(I_T)} \leq cT^{\frac{p-1}{p}}\|u\|_Y, \quad (5.40)$$

where $c = c(p) > 0$ is a constant depending only on $p$.

Recall that [15, Section 1], for $l > 0$, the parabolic Hölder space $C^{l/2}(I_T)$ is the Banach space of functions $\upsilon(x,t)$ that are continuous in $I_T$, together with all derivatives of the form $D^r_xD^s_t\upsilon$ for $2r + s < l$, and have a finite norm $\|\upsilon\|_{C^{l/2}(I_T)} = \langle \upsilon \rangle_{C^{l/2}(I_T)} + \sum_{j=0}^{[l]} \langle \upsilon \rangle^{(j)}_{C^{l/2}(I_T)}$.

Another very useful inequality in our study is a Sobolev estimate for parabolic equations. To state this estimate, we consider solutions $u \in Y$, $u = 0$ on $\partial^p(I_T)$, of:

$$u_t = \varepsilon u_{xx} + f, \quad f \in L^p(I_T) \text{ called the source term}, \quad (5.41)$$

then we have

$$\frac{\|u\|_{L^p(I_T)}}{T} + \frac{\|u_x\|_{L^p(I_T)}}{\sqrt{T}} + \|u_{xx}\|_{L^p(I_T)} + \|u_t\|_{L^p(I_T)} \leq c\|f\|_{L^p(I_T)}, \quad (5.42)$$

where $c = c(\varepsilon, p) > 0$ is a constant depending only on $p$ and $\varepsilon$.

Now, we may state the main proposition of this section:
Proposition 5.1 (Fixed point argument). Let $p > 3$ and let

$$\rho^0, \kappa^0 \in C^\infty(\overline{I}),$$

be two given functions such that $\rho^0(0) = \rho^0(1) = \kappa^0(0) = 0$ and $\kappa^0(1) = 1$. Suppose furthermore that

$$
\begin{align*}
\kappa_x^0 &\geq \gamma_0 \quad \text{on } I, \\
\|D^s_x \rho^0\|_{L^\infty(I)} &\leq M_0, \quad s = 1, 2, \\
\|D^s_x \kappa^0\|_{L^\infty(I)} &\leq M_0, \quad s = 1, 2,
\end{align*}
$$

(5.43)

with $0 < \gamma_0 \leq M_0$. Then there exists a unique solution $(\rho, \kappa) \in Y^2$ of (2.11), (2.12), and (2.13) where

$$T = T(M_0, \gamma_0, \varepsilon, p), \quad 0 < T < 1.$$ 

Moreover, this solution satisfies

$$
\begin{align*}
\gamma_0/2 \leq \kappa_x &\leq 2M_0 \quad \text{on } \overline{I_T}, \\
|\rho_x| &\leq 2M_0 \quad \text{on } \overline{I_T}. 
\end{align*}
$$

(5.44)

Proof. Throughout the proof, and in various estimates we suppose that $0 < T < 1$. This is in no way a problem since we have to choose $T$ small enough to construct our solution. The proof uses a fixed point argument on a subspace of $Y$. Looking at (5.44), we can artificially modify (2.11) using suitable truncations. To simplify our computations, we set

$$
\begin{align*}
\tilde{\rho} := \rho \kappa_0/2 + (\kappa_x - \gamma_0/2)^+ + \rho \kappa \kappa_x \\
\tilde{\kappa} := \kappa_x/2 + \gamma_0/2 + \kappa \kappa_x
\end{align*}
$$

(5.45)

with the same initial and boundary conditions (2.12) and (2.13). It is worth noticing that when (5.44) is satisfied then (5.45) coincides with (2.11). On the other hand, condition (5.44) also suggests that we consider functions $u \in Y$ of bounded gradients, i.e. $\|u_x\|_{L^p(I_T)} \leq \lambda$ where $\lambda > 0$ is a fixed sufficiently large constant. For this reason, define the spaces $Y^\rho_\lambda$ and $Y^\kappa_\lambda$ as follows:

$$
Y^\rho_\lambda = \left\{ u \in Y; \|u_x\|_{L^p(I_T)} \leq \lambda, u = \rho^0 \text{ on } \partial^\rho(I_T) \right\},
$$

and

$$
Y^\kappa_\lambda = \left\{ u \in Y; \|u_x\|_{L^p(I_T)} \leq \lambda, u = \kappa^0 \text{ on } \partial^\rho(I_T) \right\}.
$$

Define the application:

$$
\Xi : Y^\rho_\lambda \times Y^\kappa_\lambda \mapsto Y^\rho_\lambda \times Y^\kappa_\lambda
$$

$$
(\tilde{\rho}, \tilde{\kappa}) \mapsto \Xi(\tilde{\rho}, \tilde{\kappa}) = (\rho, \kappa),
$$

where $(\rho, \kappa)$ is the solution of:

$$
\begin{align*}
\kappa_t = &\varepsilon \kappa_{xx} + \frac{\rho \kappa I(\rho \kappa)}{(\gamma_0/2) + (\kappa_x - \gamma_0/2)^+} + \rho \kappa J(\kappa_x) \quad \text{on } I_T, \\
\rho_t = &\varepsilon \rho_{xx} + \tilde{\kappa}_x J(\kappa_x) \quad \text{on } I_T,
\end{align*}
$$

(5.46)
The existence and uniqueness of \((\rho, \kappa) \in Y^\rho \times Y^\kappa\) solution of (5.46) is obtained in two steps. In a first step, while having the initial and boundary conditions (2.12) and (2.13), we find a solution \(\rho \in Y\) of the second equation of (5.46), then we plug it into the first equation to get a solution \(\kappa \in Y\). Here, the existence and uniqueness of both solutions are guaranteed by [15, Theorem 9.1]. It is worth mentioning that Theorem 9.1 in [15] requires a compatibility condition of order 0 on the initial and boundary data. Those conditions are satisfied by our boundary assumptions on \(\rho^0\) and \(\kappa^0\) (see Proposition 5.1).

In a second step, we use (5.42) basically on the functions:

\[
\hat{\rho} = \rho - \rho^0 \quad \text{and} \quad \hat{\kappa} = \kappa - \kappa^0,
\]

to gain the \(L^p\) bounds on \(\rho_x\) and \(\kappa_x\) if we choose \(T\) small enough. The above steps ensure that the application \(\Xi\) is well defined at least for sufficiently small time.

We now show that \(\Xi\) is a contraction. In fact, let \(\Xi(\hat{\rho}, \hat{\kappa}) = (\rho, \kappa)\) and \(\Xi(\hat{\rho}', \hat{\kappa}') = (r, k)\). The couple \((\rho - r, \kappa - k)\) is the solution of the following system:

\[
\begin{cases}
(\kappa - k)_t = \varepsilon(\kappa - k)_{xx} + F_1 \\
(\rho - r)_t = (1 + \varepsilon)(\rho - r)_{xx} + F_2
\end{cases} \quad \text{on } I_T
\]

(5.47)

where

\[
F_1 = \frac{\rho_{xx}I(\hat{\rho}_x)}{(\gamma_0/2) + (\hat{\kappa}_x - \gamma_0/2)^+} - \frac{r_{xx}I(\hat{\kappa}_x)}{(\gamma_0/2) + (\hat{k}_x - \gamma_0/2)^+} + \rho_xJ(\hat{\kappa}_x) - r_xJ(\hat{k}_x)
\]

and

\[
F_2 = \hat{\kappa}_xJ(\hat{\kappa}_x) - \hat{k}_xJ(\hat{k}_x),
\]

with

\[
(\rho - r, \kappa - k) = (0, 0) \quad \text{on } \partial^0(I_T).
\]

In the remaining part of the proof, the constant \(c > 0\) will always denote a quantity that depends on all constants in Proposition 5.1 but independent of \(T\). Using (5.48) and (5.42), we deduce that

\[
\begin{cases}
\| \kappa - k \|_Y \leq c \| F_1 \|_{L^p(I_T)} \\
\| \rho - r \|_Y \leq c \| F_2 \|_{L^p(I_T)}
\end{cases}
\]

(5.48)

We now estimate the term \(\| \rho - r \|_Y\). In fact, since \(J\) is bounded Lipschitz continuous, we can infer that:

\[
|F_2| \leq |\hat{\kappa}_x| |J(\hat{\kappa}_x) - J(\hat{k}_x)| + |J(\hat{k}_x)||\hat{\kappa}_x - \hat{k}_x|
\]

\[
\leq \text{Lip}(J)|\hat{\kappa}_x||\hat{\kappa}_x - \hat{k}_x| + \sqrt{2M_0}|\hat{\kappa}_x - \hat{k}_x|,
\]

therefore

\[
\| F_2 \|_{L^p(I_T)} \leq c(\| \hat{\kappa}_x \|_{L^p(I_T)} + 1)(\| \hat{\kappa} - \hat{k} \|)_{L^\infty(I_T)}.
\]

By (5.40) and the fact that \(\| \hat{\kappa}_x \|_{L^p(I_T)} \leq \lambda\), we deduce that

\[
\| F_2 \|_{L^p(I_T)} \leq c(\lambda + 1)T^{-\frac{3}{2p}} \| \hat{\kappa} - \hat{k} \|_Y.
\]

(5.49)

Using the second equation of (5.48) and (5.49), we finally get:

\[
\| \rho - r \|_Y \leq cT^{-\frac{3}{2p}} \| \hat{\kappa} - \hat{k} \|_Y.
\]

(5.50)

We need to estimate also the term \(\| \kappa - k \|_Y\). We write

\[
F_1 = A_1 + A_2 + A_3 + A_4,
\]
with

\[
\begin{aligned}
A_1 &= \frac{I(\hat{\rho}_x)}{(\gamma_0/2) + (\hat{k}_x - \gamma_0/2)^+} (\rho_{xx} - r_{xx}) , \\
A_2 &= \frac{r_{xx}(I(\hat{\rho}_x) - I(\hat{r}_x))}{(\gamma_0/2) + (\hat{k}_x - \gamma_0/2)^+} , \\
A_3 &= r_{xx} I(\hat{r}_x) \left( \frac{1}{(\gamma_0/2) + (\hat{k}_x - \gamma_0/2)^+} - \frac{1}{(\gamma_0/2) + (\hat{k}_x - \gamma_0/2)^+} \right) , \\
A_4 &= \rho_x (J(\hat{k}_x) - J(\hat{r}_x)) + J(\hat{k}_x)(\rho_x - r_x).
\end{aligned}
\]  

(5.51)

We estimate the $L^p$ norms of $A_i, i = 1, 2, 3, 4$. First remark that the coefficient of $(\rho_{xx} - r_{xx})$ in $A_1$ is bounded, hence by (5.50) we deduce that:

\[
\| A_1 \|_{L^p(I_T)} \leq c T^{p-3} \| \hat{k} - \hat{\rho} \|_Y .
\]  

(5.52)

Also remark that $r$ solves the second equation of (5.46) with $\hat{k}$ replaced by $\hat{\rho}$, hence (after possibly using the Sobolev estimate (5.42) for $r = r - \rho^0$) we deduce that:

\[
\| r_{xx} \|_{L^p(I_T)} \leq c(M_0 + \lambda \sqrt{2M_0}) .
\]

Therefore, using (5.40) and the fact that $I$ is Lipschitz, we get:

\[
\| A_2 \|_{L^p(I_T)} \leq c T^{p-3} \| \hat{k} - \hat{\rho} \|_Y .
\]  

(5.53)

For $A_3$, we use the same arguments as for $A_2$, and we are lead to:

\[
\| A_3 \|_{L^p(I_T)} \leq c T^{p-3} \| \hat{k} - \hat{\rho} \|_Y .
\]  

(5.54)

Here we have used that the function $x \to \frac{1}{(\gamma_0/2)(x - \gamma_0/2)}$ is Lipschitz continuous over $\mathbb{R}$.

For $A_4$, we notice that the first term in the right hand side of it is treated as in $A_2$, while the second term is treated as in $A_1$. Hence:

\[
\| A_4 \|_{L^p(I_T)} \leq c T^{p-3} \| \hat{k} - \hat{\rho} \|_Y .
\]  

(5.55)

Using (5.52), (5.53), (5.54), and (5.55), we finally get:

\[
\| F_{\gamma} \|_{L^p(I_T)} \leq c T^{p-3} \left( \| \hat{k} - \hat{\rho} \|_Y + \| \hat{\rho} - \hat{\rho} \|_Y \right) .
\]  

(5.56)

Using the first equation of (5.48) and (5.56), we finally get:

\[
\| \kappa - k \|_Y \leq c T^{p-3} \left( \| \hat{k} - \hat{\rho} \|_Y + \| \hat{\rho} - \hat{\rho} \|_Y \right) .
\]  

(5.57)

Equations 5.50 and 5.57 show that, for $T > 0$ sufficiently small, the application $\Xi$ is a contraction, and eventually it has a unique fixed point $(\rho, \kappa)$ solution of system (5.45), (2.12), and (2.13). Lastly, to get rid of the artificial truncations $I$ and $J$ in (5.45), and to show (5.44), we use the embedding (5.39) and the initial conditions (5.43). This is again with the possibility of reducing the time $T$.

Remark 5.2. The embedding (5.39) infers that $\kappa_x \in C^{a, a/2}((\gamma_0/2)^{1/2} I_T)$ then, since $\frac{\gamma_0}{2} \leq \kappa_x \leq 2M_0$, we easily deduce:

\[
\kappa_x \sqrt{\kappa_x} \in C^{a, a/2}((\gamma_0/2)^{1/2} I_T).
\]

This simple observation suggests that we may have a better regularity for the solution $\rho$ obtained in Proposition 5.1, which, by its turn, can also lead to a better regularity on $\kappa$ due to the coupling in (2.11).
Proposition 5.3 (Bootstrap argument). Under the same hypothesis of Proposition 5.1 and if, in addition, the functions \( \rho^0, \kappa^0 \) satisfy the condition:

\[
\begin{aligned}
(1 + \varepsilon) \rho^0_{xx} + \kappa^0 \sqrt{\kappa^0} &= 0 & \text{on } \partial I, \\
(1 + \varepsilon) \kappa^0_{xx} + \rho^0 \sqrt{\kappa^0} &= 0 & \text{on } \partial I, 
\end{aligned}
\]  

(5.58)

then the solution \((\rho, \kappa)\) obtained in Proposition 5.1 satisfies:

\[
(\rho, \kappa) \in (C^{3+\alpha/2, \beta/2}(\overline{I_T}) \cap C^\infty(I \times (0, T)))^2. 
\]  

(5.59)

Remark 5.4. The assumption (3.15) on the initial data, together with the constant boundary values, define a compatibility condition of order 1. In other words, we get:

\[
\begin{aligned}
0 &= \kappa_x(x, 0) = \varepsilon \kappa_{xx}(x, 0) + \rho_x(x, 0) \rho_{xx}(x, 0) / \kappa_x(x, 0) + \rho_x(x, 0) \sqrt{\kappa_x(x, 0)}, & \text{for } x \in \partial I, \\
0 &= \rho_t(x, 0) = (1 + \varepsilon) \rho_{xx}(x, 0) + \kappa_x(x, 0) \sqrt{\kappa_x(x, 0)}, & \text{for } x \in \partial I.
\end{aligned}
\]

This boundary condition let us conclude that (see [15, Theorem 5.2]), if the source terms (see equation 5.41) in (2.11) is of class \( C^{\beta, \beta/2}(\overline{I_T}) \) for \( 0 < \beta < 2 \), then the solution \((\rho, \kappa)\) will be of class \( C^{2+\beta, 2\beta/2}(\overline{I_T}) \).

Proof of Proposition 5.3. The proof follows the idea of Remark 5.2. In fact, since \( \kappa_x \in C^{a, a/2}(\overline{I_T}) \) with \( \frac{a}{2} \leq \kappa_x \leq 2M_0 \), then the continuous function \( \kappa_x \sqrt{\kappa_x} \) satisfies:

\[
\begin{align*}
|\kappa_x(x, t) \sqrt{\kappa_x(x, t)} - \kappa_x(y, t) \sqrt{\kappa_x(y, t)}| & \leq |\kappa_x(x, t)| |\sqrt{\kappa_x(x, t)} - \sqrt{\kappa_x(y, t)}| + |\sqrt{\kappa_x(y, t)}| |\kappa_x(x, t) - \kappa_x(y, t)|, \\
& \leq \frac{\sqrt{2}M_0}{\sqrt{\gamma}} |\kappa_x(x, t) - \kappa_x(y, t)| + \sqrt{2}M_0 |\kappa_x(x, t) - \kappa_x(y, t)|, \\
& \leq c|x - y|^a.
\end{align*}
\]

Similarly, we obtain:

\[
|\kappa_x(x, t) \sqrt{\kappa_x(x, t)} - \kappa_x(x, s) \sqrt{\kappa_x(x, s)}| \leq c|t - s|^{a/2}.
\]

These arguments show that the source term \( \kappa_x \sqrt{\kappa_x} \) of the second equation of (2.11) is of class \( C^{a, a/2}(\overline{I_T}) \). Remark 5.4 with \( 0 < \beta = a < 2 \) and the compatibility conditions (5.58), ensures that \( \rho \in C^{a+2, a+\alpha/2}(\overline{I_T}) \), which, by its turn, and thanks to similar computations as above, shows that the the source term \( \frac{\partial \rho_x}{\partial s} + \rho_x \sqrt{\kappa_x} \) of the first equation of (2.11) is of class \( C^{a, a/2}(\overline{I_T}) \).

Finally, we also deduce that \( \kappa \in C^{a+2, a+\alpha/2}(\overline{I_T}) \) and then \( \kappa_x \in C^{1+a, a/2}(\overline{I_T}) \).

We now repeat exactly the same ideas but with this new regularity on \( \kappa_x \), taking into consideration that still \( 0 < \beta = a + 1 < 2 \), and hence Remark 5.4 is still applicable. Therefore, we are finally lead to \((\rho, \kappa) \in (C^{3+\alpha, 1+\alpha/2}(\overline{I_T}))^2\). It is worth noticing that this is the maximum parabolic Hölder regularity that could be obtained up to the boundary since the compatibility condition (5.58) is not sufficient when \( \beta > 2 \). Indeed, higher order compatibility assumptions are required if we to achieve more regularity on the boundary.
To get the interior $C^\infty$ regularity, we need to carefully overcome the problem of compatibility at $t = 0, x \in \partial I$. Indeed, let $0 < \delta < T$, and define any test function $\phi_\delta \in C^\infty[0, T]$ by:

$$\phi_\delta(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{\delta}{3}, \\ \phi_\delta(t) \in (0, 1) & \text{if } \frac{\delta}{3} \leq t \leq \frac{2\delta}{3}, \\ 1 & \text{if } \frac{2\delta}{3} \leq t \leq T. \end{cases} \quad (5.60)$$

By introducing $\bar{\rho} = \rho \phi_\delta$ and $\bar{\kappa} = \kappa \phi_\delta$, we can easily see that $(\bar{r}, \bar{\kappa})$ satisfy a parabolic system where higher order compatibility conditions on the initial data are satisfied. Hence, the parabolic Hölder regularity can be infinitely applied for $\bar{\rho}$ and $\bar{\kappa}$. Accordingly

$$(\bar{\rho}, \bar{\kappa}) \in C^\infty(I_T).$$

From (5.60) and (5.61) we deduce that

$$(\rho, \kappa) = (\bar{\rho}, \bar{\kappa}) \text{ on } \left[\frac{2\delta}{3}, T\right], \quad \forall 0 < \delta < T.$$ 

Thus $(\rho, \kappa) \in (C^\infty(\bar{I} \times (0, T)))^2$ and the proof follows.

\section{Conclusion and Open Problems}

This existence result of this paper is, up to our knowledge, the first one dealing with the model of GCZ\cite{8} with the physical exterior stress. It differs from that obtained in [13] where the stress was assumed to be a constant which somehow simplifies the analysis on the parabolic estimates. It is still an open problem, however, how to obtain the long time existence of the solution.

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\section{References}


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