Traveling waves for a model of individual clustering with logistic growth rate

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We consider a traveling wave problem for a one-dimensional model of individual clustering or aggregation. This model, originally formulated by Grindrod [J. Math. Biol. 26(6), 651–660 (1988)], describes the mechanism of individual dispersion when individuals are able to utilize information about their local environment. Specifically, it assumes that each individual disperses randomly with probability $\delta$ and disperses deterministically so as to improve his reproductive rate with probability $1 - \delta$. In this paper, we prove the existence of a traveling wave solution when the probability of random dispersion exceeds a critical value $\delta^*$ uniquely determined by the biased individual velocity. Published by AIP Publishing.

I. PHYSICAL MOTIVATION AND MAIN RESULTS

We consider the model of individual aggregation introduced by Grindrod. This model describes the evolution of a biological population density where individuals are partially aware to their local environment and consequently are assumed to experience both unconditional and conditional dispersion. Unconditional dispersion refers to dispersion disregarding the environment or the presence of other organisms (e.g., pure diffusion and random motion). However, conditional dispersion refers to that which is influenced by the environment and in response to the threat of other organisms or other externally controlled factors.

In reality, species are neither totally blind to the surrounding habitat nor will their movement perfectly track all resource gradients. It is more expected that their movements are a combination of both random and biased ones. Diffusion models incorporating directed movement upward along resource gradients have also been considered in Refs. 1, 3, 4, and 6.

A. The mathematical model

We now derive the mathematical equations corresponding to the model suggested by Grindrod (for details, see Ref. 9, Sec. 1). Let $u = u(x, t)$ denote the density of the population at location $x \in \mathbb{R}^n$ and time $t$, and let $E = E(u, x, t)$ denote the projected net growth rate per individual at location $x$ and time $t$, within a local population density $u$. The dispersal of the species can be described in terms of the flux associated with their motion,

$$\Phi_u = -\delta \nabla u + (1 - \delta)uw,$$

where $w$ stands for the average velocity of those individuals dispersing so as to improve their projected net reproductive rates,

$$w \approx \lambda \nabla E, \quad \lambda > 0. \quad (1.1)$$

In other terms, $w$ is the convection speed that is responsible for aggregation. Here $\delta \in (0, 1)$ represents the probability of random diffusion and $1 - \delta$ represents the probability of the directed movement upward along the environmental gradient $E$. The population balance thus reads

$$u_t = -\nabla \Phi_u + uE(u, x, t). \quad (1.2)$$

By slightly modifying Eq. (1.1), we set

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\[-\varepsilon \Delta w + w = \lambda \nabla E,\]  

(1.3)

where \(\varepsilon > 0\) is a small constant. The role of the first term in (1.3) is to smooth the convection speed in order to avoid possible anti-diffusion so that \(w\) is really a local average of the optimal velocity \(\lambda \nabla E\). Putting (1.2) and (1.3) together, we have

\[
\begin{aligned}
&\frac{\partial u}{\partial t} - \delta \frac{\partial u}{\partial x} + (1 - \delta) \nabla (uw) = uE(u, x, t), \\
&-\varepsilon \Delta w + w = \lambda \nabla E(u, x, t),
\end{aligned}
\]

(1.4)

\(\text{holding for } t > 0 \text{ and } x \text{ in some domain of } \mathbb{R}^a.\)

To reflect the homogeneity of environment, it is assumed that the projected net rate of reproduction \(E\) is independent of \(x\) and \(t\). In some sense, we may think that the quality and the quantity of resources depend solely on the density of the population. In particular, we are interested in the logistic growth rate,

\[E(u) = 1 - u.\]

One may then easily notice, using (1.1), that \(w \approx -\lambda \nabla u\) and hence the individuals dispersing so as to maximise \(E\) would seek isolation. Finally, we consider the one-dimensional problem associated with (1.4),

\[
\begin{aligned}
&\frac{\partial u}{\partial t} - \delta \frac{\partial u}{\partial x} + (1 - \delta) \nabla (uw) = u(1 - u), \quad x \in \mathbb{R}, \ t > 0, \\
&-\varepsilon \frac{\partial^2 w}{\partial x^2} + w = -\lambda \frac{\partial u}{\partial x}, \quad x \in \mathbb{R}, \ t > 0,
\end{aligned}
\]

(1.5)

where the notation \(u_t\), or \(u_x\) means time or space derivatives. We complete the system by imposing the following boundary conditions on \(u\) and \(w\):

\[u(-\infty) = 1, \ u(+\infty) = 0, \quad \text{and} \quad w(-\infty) = w(+\infty) = 0.\]

(1.6)

B. Main result

Mathematically, system (1.4) couples a convection-diffusion equation in \(u\) together with an elliptic equation in \(w\). It has been widely studied in Refs. 14–16 in the case of a bounded domain \(\Omega \subset \mathbb{R}^a, \ n = 1, 2\), supplemented with the no-flux boundary conditions

\[n \cdot \nabla u = n \cdot w = 0 \quad \text{and} \quad \partial_n w \times n = 0,
\]

where the latter boundary condition is useless in the one-dimensional case. In Ref. 16, for \(E \in C^2(\mathbb{R}^a)\) and for a suitable positive initial value \(u_0 \in W^{1,p}(\Omega), \ p > n \geq 1\), the author shows the existence of a positive, smooth, and maximal solution \(u \in C \left([0, T_{\text{max}}), W^{1,p}(\Omega)\right) \cap C \left((0, T_{\text{max}}), W^{2,p}(\Omega)\right)\). Moreover, the author studies (see Refs. 15 and 16) the global existence issue in one- and two-dimensional cases, for the two specific forms of \(E\) as suggested in Ref. 9: the bistable case where \(E(u) = (1 - u)(a - u)\) for some \(a \in (0, 1)\) and the monostable (logistic) case where \(E(u) = 1 - u\). It is worth mentioning that in the logistic case when \(\Omega = (-1, 1)\), the author also examines the asymptotic behaviour of solutions for large \(t\) and the convergence of \(u\) to a steady state in \(L^2(-1, 1)\). The study of the logistic case is completed by the limiting behaviour as \(\varepsilon \to 0\). In Ref. 14, the author is interested in the case where the aggregation mechanism is dominant in both bistable and logistic cases. The author shows, by a compactness argument, that the solution \((u, w)\) converges to a unique strong solution of the corresponding transport problem.

In this paper, we consider system (1.4) in the one-dimensional case with a logistic growth. In particular, we are concerned with the existence of traveling waves of (1.5) and (1.6). The traveling wave solutions moving with a speed \(c\) are special solutions having the forms \(u(x - ct)\) and \(w(x - ct)\) that satisfy

\[
\begin{aligned}
&-cu' - \delta u'' + (1 - \delta)(uw)' = u(1 - u),
\\
&-\varepsilon w'' + w = -\lambda u',
\end{aligned}
\]

(1.7)

subjected to boundary conditions (1.6). We prove the following result:
Theorem 1.1. Let $\varepsilon, \lambda > 0$ be positive constants and define $\delta_* = \frac{\lambda}{\varepsilon^2}$. Then, for every
$$0 < \delta < \delta_* < 1,$$
there exists a traveling wave solution $(u, w, c_\ast)$ of (1.7) with boundary conditions (1.6).

This theorem ensures the existence of a critical probability value $0 < \delta < 1$, where we are able to construct traveling wave solutions if the probability of random diffusion $\delta$ exceeds $\delta_*$. However, the question of existence or nonexistence of traveling wave solutions for $\delta \leq \delta_*$ remains open.

When $\lambda \to +\infty$, our limiting solution will be a member of the family of the traveling waves of the Kolmogorov-Petrovskii-Piskunov (KPP) equation with the Fisher birth term $u(1 - u)$, obtained for $\delta = 1$ (see Refs. 7 and 12).

In order to prove Theorem 1.1, we first consider an integrated system of (1.7) that reads
$$\begin{cases}
-cu' - \delta u'' - (1 - \delta)(u'v)' = u(1 - u), \\
-\varepsilon v'' + v = \lambda u,
\end{cases}$$
subjected with the slightly different boundary conditions
$$u(-\infty) = 1, \ u(+\infty) = 0 \quad \text{and} \quad v(-\infty) = \lambda, \ v(+\infty) = 0.$$  
(1.10)
This system can be derived from models like (1.4) by formally taking $w = -\nabla v$. Such systems appear in the study of the biomechanical process known as chemotaxis, where $u$ represents the density of living organisms (such as bacteria) while $v$ represents the concentration of some chemical that could induce the movement of these organisms. To be more precise, system (1.9) describes a chemorepulsion motion due to the negative coefficient $-(1 - \delta)$ in front of the term $\nabla(u \nabla v)$ involving the chemical concentration. Chemorepulsion is a biophysical process where the regions of high chemical concentration have a repulsive effect on the organisms (see Ref. 5 for the existence of solutions of a chemorepulsion system).

The reason behind introducing systems (1.9) and (1.10) is two-fold; first, having a sufficiently regular solution $(u, v)$ of (1.9) and (1.10) [say $u, v \in W^{2,\infty}(\mathbb{R})$, for instance], we may directly obtain a solution $(u, w)$ of (1.7) and (1.6) when taking $w = -v'$ and keeping $u$. Second, we notice that system (1.9) can be decoupled into one KPP equation, with a nonlocal drift, by writing the second equation as a convolution $v = \lambda K_{\infty} \ast u$. This, together with the fact that we can directly import $v''$ into the first equation, serves to apply a maximum principle in order to obtain a priori estimates on $u$. Indeed, we prove the following theorem of independent interest:

Theorem 1.2. Under the same assumptions of Theorem 1.1, there exists a solution $(u, v, c_\ast)$ of (1.9) with boundary conditions (1.10).

We note that Theorem 1.1 will be a direct consequence of Theorem 1.2 once we can obtain sufficient regularity on $v$. All solutions in Theorem 1.1 and Theorem 1.2 are assumed to be in the space $W^{2,\infty}(\mathbb{R})$.

Other situations where traveling waves appear in the literature for KPP equations with the Fisher term can be found in Refs. 7 and 12 as mentioned above. The authors show the existence of a half-line $[c_\ast, +\infty]$ of speeds associated with traveling waves in the logistic case and the existence of unique monotone traveling waves in the bistable case. We can also refer to Ref. 10 for the existence of traveling waves for chemotaxis systems. Other models related to chemotraction or chemorepulsion have been studied in Ref. 13 for a logistic growth rate and in Ref. 8 for a bistable growth rate.

The strategy of the proof of Theorem 1.2 relies on introducing a smooth monotonic cutoff function $g_0$ such that $g_0(u) = 0$ for $u \leq 1$ and $g_0(u) = 1$ for $u \geq 2$, where we set
$$g(u) = g_0 \left( \frac{u}{\theta_0} \right).$$
(1.11)
We then consider a regularized system
$$\begin{cases}
-cu' - \delta u'' - (1 - \delta)(g(u)uv)' = g(u)u(1 - u), \\
-\varepsilon v'' + v = \lambda u,
\end{cases}$$
(1.12)
with the same boundary conditions (1.10). Physically speaking, the cutoff means that the organisms feel the chemorepellent and reproduce only if their density exceeds a critical value $\theta_0$. Mathematically, the cutoff plays a very similar role as the ignition temperature in combustion theory.$^{2,11}$

The first step is to construct a solution $(u(x; \theta_0), v(x; \theta_0), c(\theta_0))$ of (1.12) for sufficiently small $\theta_0$ and to obtain some uniform bounds independent of $\theta_0$. This is obtained by adapting the method of Refs. 2 and 11 where we first build up solutions $(u_0(x; \theta_0), v_0(x; \theta_0), c_0(\theta_0))$ on bounded domains $[-a, a]$ transforming the problem to a fixed point equation that we can solve with the Leray-Schauder degree theory. We thereafter let $a \to +\infty$ and extract a subsequence of $(u_0(x; \theta_0), v_0(x; \theta_0), c_0(\theta_0))$ converging to one solution of (1.12) and (1.10). The proof is based mainly on qualitative properties of $(u_0(x; \theta_0), v_0(x; \theta_0), c_0(\theta_0))$ and a priori estimates independent of $a$.

The second step is to get similar qualitative properties and a priori estimates on $(u(x; \theta_0), v(x; \theta_0), c(\theta_0))$ that allows us to let $\theta_0 \to 0$ and obtain a traveling wave solution $(u, v, c_*)$ of (1.9) and (1.10) without the cutoff as stated in Theorem 1.2. Finally, as mentioned above, Theorem 1.1 thus follows by improving the regularity on $(u, v, c_*)$ mainly using the elliptic equation satisfied by $v$.

Concerning the regularised system (1.12), we prove the following proposition:

**Proposition 1.3.** Let $\theta_0 > 0$ be sufficiently small. Under the same assumptions of Theorem 1.1, there exists a solution $(u(x; \theta_0), v(x; \theta_0), c(\theta_0))$ of (1.12) and (1.10). Moreover, there exists a constant $K > 0$ (independent of $\theta_0$) such that the following uniform bounds hold:

$$
\begin{align*}
0 < u(x; \theta_0), v(x; \theta_0) < 1, & \quad 0 < 1/K \leq c(\theta_0) \leq K < +\infty, \\
\int g(u(x; \theta_0))u(x; \theta_0)(1 - u(x; \theta_0))^2 dx + \int |u'(x; \theta_0)|^2 dx + \int |v'(x; \theta_0)|^2 dx \leq K.
\end{align*}
$$

(1.13)

The organisation of this paper is as follows. In Sec. II, we treat problems (1.12) and (1.10) on a bounded interval $[-a, a]$. We prove the existence by a homotopy argument and we obtain the main estimates independent of $a$. We let $a \to +\infty$ in Sec. III, thus showing Proposition 1.3. Finally, in Sec. IV, we let $\theta_0 \to 0$ and we prove Theorem 1.2. Consequently, the regularity of the solution obtained in Theorem 1.2 permits to show Theorem 1.1.

In the remaining part of this paper, the terms $C$ and $K$ denote the positive constants that could vary from line to line but only depend on $\epsilon, \delta$, and $\lambda$.

## II. EXISTENCE ON A BOUNDED DOMAIN $[-a, a]$

In order to prove Proposition 1.3, we consider a problem similar to (1.12), but posed on a bounded domain $[-a, a]$. This problem is then equivalent to a fixed point equation, which we can solve by the Leray-Schauder degree theory. Our approach follows traditional methods (see Refs. 2 and 11, for instance) that we adapt to our situation. The main difficulty is to obtain the necessary a priori estimates that will allow first to let $a \to +\infty$ (see Sec. III) and, in a later step (see Sec. IV), to let $\theta_0 \to 0$. Other difficulties arise in showing upper and lower bounds on $c$ and in proving that the states $u = 1, u = 0$ are reached at infinity.

We consider the following boundary value problem posed on $[-a, a]$: Find the triplet $(u_a, v_a, c_a)$ (we drop the term $\theta_0$ from the notation) satisfying

$$
\begin{align*}
-c_a u_a'' - \delta u_a'' - (1 - \delta)(g(u_a)u_a v_a')' &= g(u_a)u_a(1 - u_a), \\
-\epsilon v_a'' + v_a &= \lambda u_a,
\end{align*}
$$

(2.1)

with the boundary conditions for $u_a$ are

$$
u_a(-a) = 1, \quad u_a(a) = 0,
$$

(2.2)

and

$$v_a = \lambda K_\epsilon \ast \tilde{u}_a, \quad K_\epsilon(x) = \frac{e^{-|x|/\sqrt{\epsilon}}}{2\sqrt{\epsilon}},
$$

(2.3)
where \( \bar{u}_a \) is the extension of \( u_a \) to the whole real line,

\[
\bar{u}_a(x) = \begin{cases} 
1, & x < -a, \\
u_a(x), & -a \leq x \leq -a, \\
0, & x > a.
\end{cases}
\]  

We note that conditions (2.3) and (2.4) act as a replacement of imposing boundary conditions on \( v_a \) at \( \pm a \). We also note that the function \( v_a \) is defined on \( \mathbb{R} \) and satisfies

\[
-\varepsilon v_a'' + v_a = \lambda u_a, \quad v_a(-\infty) = \lambda, \quad v_a(+\infty) = 0.
\]  

The following estimates are due to the representation formula (2.3):

\[
\begin{cases}
\|v_a\|_{\infty} \leq \frac{\lambda}{\sqrt{\varepsilon}}\|u_a\|_{\infty}, \\
\|v_a'\|_{\infty} \leq \frac{\lambda}{\sqrt{\varepsilon}}\|u_a\|_{\infty}, \\
\|v_a''\|_{\infty} \leq \frac{C\varepsilon}{\lambda}\|u_a\|_{\infty}.
\end{cases}
\]  

We normalise the function \( u_a \) so that

\[
\max_{x \geq 0} u_a(x) = \theta_0. 
\]  

Hence, from the maximum principle, we directly obtain \( u_a(0) = \theta_0 \) and thus, using (2.1), \( u_a \) satisfies the boundary value problem

\[
-c_au_a'(x) - \delta u_a''(x) = 0, \quad 0 \leq x \leq a, \quad u_a(0) = \theta_0, \quad u_a(a) = 0.
\]  

We can then find \( u_a \) explicitly on the interval \([0, a]\),

\[
u_a(x) = \theta_0 \frac{e^{-\frac{\delta}{\varepsilon}x} - e^{-\frac{\varepsilon}{\delta}a}}{1 - e^{-\frac{\varepsilon}{\delta}a}}.
\]  

**Proposition 2.1.** Under assumption (1.8), there exists a solution \((u_a, v_a, c_a)\) of (2.1)–(2.3) and (2.7) satisfying uniform bounds (1.13).

The remaining part of this section is devoted to the proof of this proposition using a homotopy argument.

Denote by \( I_a = (-a, a) \) and set by definition

\[
X = C^{1,\alpha}(I_a) \times C^{1,\alpha}(I_a) \times \mathbb{R}.
\]  

This is a Banach space equipped with norm \( \| (u, v, c) \|_X = \max \{ \| u \|_{C^{1,\alpha}(I_a)}, \| v \|_{C^{1,\alpha}(I_a)}, |c| \} \). Let us consider the mapping sending each element \((u, v, c)\) of \( X \) to the unique solution of the following linear system, indexed by the parameter \( 0 \leq \tau \leq 1 \):

\[
\begin{cases}
-cU' - \delta U'' - \tau(1 - \delta)(g(u)Uv')' = \tau g(u)u(1 - u), \\
-\varepsilon V'' + V = \tau\lambda u, \\
U(-a) = 1, \quad U(a) = 0, \\
V(x) = \tau\lambda K_e \ast \bar{u},
\end{cases}
\]  

where \( \bar{u} \) is the extension of \( u \) as in (2.4). By standard elliptic regularity, we can easily check that \( U \) and \( V \) are in the Sobolev space \( H^2(I_a) \) and even in \( W^{2,\alpha}(I_a) \). Since the embedding \( H^2(I_a) \subseteq C^{1,\alpha}(I_a) \) is compact, we can define a compact mapping, indexed by \( \tau \), from \( X \) to \( X \),

\[
K_{\tau} : (u, v, c) \rightarrow (U, V, \theta_0 - \max_{x \geq 0} u + c).
\]  

Similarly, the mapping \( K : X \times [0, 1] \rightarrow X \) defined by \((u, v, c, \tau) \rightarrow K_{\tau}(u, v, c)\) is compact and uniformly continuous with respect to \( \tau \). We then notice that every solution of (2.1)–(2.3) and (2.7) is a fixed point of \( K_{\tau} \) and vice versa: a fixed point of \( K_{\tau} \) provides a solution. Therefore, it suffices to show that operator

\[
F_{\tau} = I - K_{\tau}
\]
has a nontrivial kernel. All the above arguments permit to apply the Leray-Schauder topological degree. Indeed, thanks to the property of $K_r$, it suffices then to compute the degree of $F_r$ and to prove that this degree is well defined and nonzero. Let us introduce the open bounded set $\Omega \subset X$,

$$\Omega = \{ (u, v, c) \in X, \|u\|_{C^{1,\alpha}(\partial \Omega)} < M, \|v\|_{C^{1,\alpha}(\partial \Omega)} < M, \zeta < c < \bar{c}\},$$

(2.15)

where $M > 0$ and $-\infty < \zeta < \bar{c} < +\infty$. The degree $\text{deg}(F_r, \Omega, 0)$ of $F_r$ in $\bar{\Omega}$ at 0 will be defined if there exists $M, \zeta, \bar{c}$ such that $F_r(\partial \Omega) \neq 0$. This is demonstrated in Lemmas 2.2 and 2.3. We finally give the proof of Proposition 2.1 by computing the degree of $F_r$ in $\bar{\Omega}$ at 0, using both the invariance by homotopy and the multiplicative property of the degree.

### A. Uniform bounds and homotopy argument

Since we are going to use a homotopy argument, we introduce a family of problems depending on the homotopy parameter $0 \leq \tau \leq 1$,

$$\begin{cases} -c_{\tau,a}u''_{\tau,a} - \delta u''_{\tau,a} - \tau(1 - \delta)(g(u_{\tau,a})u_{\tau,a}v'_{\tau,a})' = \tau g(u_{\tau,a})u_{\tau,a}(1 - u_{\tau,a}), \\ -ev''_{\tau,a} + v_{\tau,a} = \tau u_{\tau,a}, \end{cases}$$

(2.16)

together with conditions (2.2),(2.3)(with the right side multiplied by $\tau$), and (2.7). It is worth noticing that in this case we get all the estimates (2.6) again with the right side multiplied by $\tau$. To simplify the notation, let us drop the subscripts $\tau$ and $a$ below. We begin by showing an upper bound for the speed $c$.

**Lemma 2.2.** Any solution of (2.2), (2.3), (2.7), and (2.16) satisfies

$$0 \leq u \leq 1, \quad 0 \leq v \leq \lambda, \quad \|v''\|_{\infty} \leq C.$$  

(2.17)

Moreover, there exists $a_0 = a_0(\theta_0) > 0$ and $K > 0$ such that for all $a > a_0$ we have

$$c \leq 2 + K.$$  

(2.18)

**Proof.** We first infer that as $g(u) = 0$ for $u \leq 0$, then $u$ cannot attain an interior negative minimum on $(-a, a)$ thus $u \geq 0$ and, using (2.3), we also get $v \geq 0$. We rewrite the first equation of (2.16) as

$$-cu'' - \delta u'' - \tau(1 - \delta)g(u)uu'v' - \tau(1 - \delta)g(u)u'v' = \tau g(u)u(1 - u) + \tau(1 - \delta)g(u)uv''$$

$$= \tau g(u)u \left(1 - u + (1 - \delta) v''\right) = \tau g(u)u \left(1 - u + (1 - \delta) \left(\frac{v}{\lambda} - \frac{\tau \lambda u}{\delta}\right)\right)$$

$$\leq \tau g(u)u \left(1 - u + \frac{(1 - \delta) \tau \lambda u}{\delta} (\|u\|_{\infty} - u)\right),$$

(2.19)

where for the second line, we have used the second equation in $v$ of (2.16), and for the third line, we have used (2.6). We claim that $u \leq 1$. In fact, if this is not true, then the function $u$ attains [see (2.2)] its maximum value $u(x_0) = \|u\|_{\infty} > 1$ at some interior point $x_0$ in $(-a, a)$. We conclude that the constant function $\bar{u} = u(x_0) = \|u\|_{\infty}$ is a super-solution of (2.19), and consequently $u \leq \bar{u}$ which leads to a contradiction as $u(a) = 0$. All the above arguments, together with (2.6), lead to the assertion (2.17).

Next, we now show that $c$ is uniformly bounded from above. Since $0 \leq u \leq \|u\|_{\infty} = 1, 0 \leq \delta, \tau, g \leq 1$, we use again inequality (2.19) to obtain

$$-cu' - \delta u'' - \tau(1 - \delta)(g(u) + g'(u)u')u' \leq \left(1 + \frac{\lambda}{\delta}\right)u.$$  

(2.20)

We set $\psi_m(x) = me^{-x}$. Then, the function $\psi_m$ satisfies

$$-c\psi''_m - \delta \psi'_m - \tau(1 - \delta)(g(u) + g'(u)u')v'_m \psi'_m = \left(c - \delta + \tau(1 - \delta) (g(u) + g'(u)u') \psi_m \right) \geq (c - 1 - K_0)\psi_m,$$

where $K_0 > 0$ is a positive constant, independent of $a$, $\tau$, and $\theta_0$, chosen so that [see (2.6)]

$$\|\tau(1 - \delta)(g(u) + g'(u)u')\|_{\infty} \leq C\|g(\sigma) + g'(\sigma)\sigma\|_{\infty} \leq K_0.$$  

(2.21)
This is possible due to the boundedness of \( g \) and the fact that \( g'(\sigma) = 0 \) for \( \sigma \notin (\theta_0, 2\theta_0) \), while for \( \sigma \in (\theta_0, 2\theta_0) \), we have
\[
|g'(\sigma)\sigma| = \left| \frac{\sigma}{\theta_0} g'(\frac{\sigma}{\theta_0}) \right| \leq 2 \max_{1 \leq x \leq 2} |g'(x)|.
\]
Suppose by contradiction that
\[
c > 2 + K_0 + \frac{\lambda}{e}.
\]
(2.22)
Then \( \psi_m \) satisfies
\[
-c\psi'_m - \delta \psi''_m - \tau (1 - \delta) (g(u) + g'(u)u) v' \psi'_m \geq \left( 1 + \frac{\lambda}{e} \right) \psi_m.
\]
Hence, in other words, the function \( \psi_m \) is a super-solution of (2.20). As \( u \leq 1 \), we deduce that for \( m > e^a \), we have
\[
\psi_m(x) = \psi_m(0) = m_0 \geq 1 \quad \text{for} \quad -a \leq x \leq a.
\]
Therefore, we may define
\[
m_0 = \inf \{ m; \psi_m > u \ \text{on} \ [-a, a] \} > 0.
\]
It is easily seen that \( \psi_m \geq u \) on \(-a, a\) and there exists \( x_1 \in [-a, a] \) such that \( \psi_m(x_1) = u(x_1) \). By the maximum principle, we know that \( \psi_m \) is a super-solution of (2.20) which is impossible as \( \psi_m(0) = 0 = u(0) \) thus also showing \( x_1 \neq 0 \). The only possibility left is that \( x_1 = a \). In this case, we obtain \( m_0 = e^{-a} \), hence
\[
\theta_0 \equiv u(0) \leq \psi_m(0) = m_0 = e^{-a},
\]
which leads to a contradiction if we choose \( a \) large enough. We finally conclude that (2.22) is not true and thus (2.18) holds true with \( K = K_0 + \frac{\lambda}{e} \).

We now turn to the lower bound for the traveling speed \( c \).

**Lemma 2.3.** With assumption (1.8), let \((u, v, c)\) be a solution of (2.2), (2.3), (2.7), and (2.16). Then there exists \( a_0 = a_0(\theta_0) > 0 \) and \( K > 0 \) such that for all \( a > a_0 \) and \( 0 < \theta_0 < 1/3 \) we have
\[
c \geq \min(\Gamma, \Gamma/5) \quad \text{with} \quad \Gamma = K \sqrt{\frac{\tau}{a}} - \frac{6\theta_0}{a},
\]
\[
\tau \int_{-a}^{a} g(u)u(1-u)^2 \, dx + \int_{-a}^{a} |u'(x)|^2 \, dx + \int_{-a}^{a} |v'(x)|^2 \, dx \leq K.
\]
Using (2.11), (2.2), and (2.7), we obtain
\[
c - \delta u'(a) + \delta u'(-a) + \tau (1 - \delta) v'(-a) = \tau \int_{-a}^{a} g(u)u(1-u) \, dx.
\]
(2.25)
We now multiply the first equation of (2.16) by \( u \) and again integrate on \([-a, a]\),
\[
\frac{c}{2} + \delta \int_{-a}^{a} |u'|^2 \, dx + \delta u'(-a) + \tau (1 - \delta) \int_{-a}^{a} g(u)uu'v' \, dx + \tau (1 - \delta) v'(-a) = \tau \int_{-a}^{a} g(u)u(1-u)^2 \, dx.
\]
(2.26)
Equations (2.25) and (2.26) give
\[
\frac{c}{2} - \delta u'(a) - \delta \int_{-a}^{a} |u'|^2 \, dx - \tau (1 - \delta) \int_{-a}^{a} g(u)uu'v' \, dx = \tau \int_{-a}^{a} g(u)u(1-u)^2 \, dx.
\]
(2.27)
Using the representation (2.3) for \( v \) (with the right side multiplied by \( \tau \)), we have
\[
v' = \tau \lambda K_0 \ast \tilde{u}'
\]
and hence
\[
\|v'\|_2 \leq \tau \lambda \|\tilde{u}'\|_2 \leq \tau \lambda \|u'\|_2.
\]
(2.28)
Using this together with the Hölder inequality, we can infer that
\[ \left| \int_{-\alpha}^{\alpha} g(u)uu' \, dx \right| \leq \tau \int_{-\alpha}^{\alpha} |u'|^2 \, dx. \]
Consequently, from (2.27) and the fact that $0 \leq \tau, u, g \leq 1$, we deduce that
\[ \tau \int_{-\alpha}^{\alpha} g(u)(1 - u)^2 \, dx + \beta \int_{-\alpha}^{\alpha} |u'|^2 \, dx \leq \frac{c}{2} + \delta |u'(a)|, \tag{2.29} \]
where, by (1.8),
\[ \beta := \delta - (1 - \delta)\lambda > 0. \tag{2.30} \]
In order to control the term $u'(a)$ appearing in the above equation, we return back to the explicit expression (2.8) of $u$. Indeed, we have
\[ u'(a) = -\frac{c \theta_0 e^{-\frac{a}{\theta_0}}}{\delta (1 - e^{-\frac{a}{\theta_0}})}, \]
therefore
\[ |u'(a)| \leq \frac{\theta_0}{a} + \frac{|c|}{\delta} \theta_0. \]
Since $\delta \leq 1$, we use the previous inequality in (2.29) to deduce that
\[ \tau \int_{-\alpha}^{\alpha} g(u)(1 - u)^2 \, dx + \beta \int_{-\alpha}^{\alpha} |u'|^2 \, dx \leq \frac{c}{2} + \frac{\theta_0}{a} + |c|\theta_0. \tag{2.31} \]
Moreover, as $u(-a) = 1$ and $u(0) = \theta_0$, there exists a constant $K_1 > 0$ such that
\[ \left( \int_{-\alpha}^{\alpha} g(u)(1 - u)^2 \, dx \right) \left( \int_{-\alpha}^{\alpha} |u'|^2 \, dx \right) \geq K_1. \tag{2.32} \]
To show (2.32), it suffices to integrate from $-\alpha$ to 0 and to consider the case where $u$ decreases from 1 to $\theta_0 < 1$. The general case can be easily deduced upon integrating over $[u \leq 0]$. Indeed, by Hölder’s inequality, we have
\[ \int_{-\alpha}^{0} \sqrt{g(u)} |1 - u||u'| \, dx \leq \left( \int_{-\alpha}^{0} g(u)(1 - u)^2 \, dx \right)^{1/2} \left( \int_{-\alpha}^{0} |u'|^2 \, dx \right)^{1/2}, \]
and by letting $v = u(x)$ and noticing that $0 \leq u \leq 1$, we get
\[ \int_{-\alpha}^{0} \sqrt{g(u)} |1 - u||u'| \, dx = -\int_{-\alpha}^{0} \sqrt{g(u)} (1 - u)u' \, dx = \int_{\theta_0}^{1} \sqrt{g(v)} (1 - v) \, dv. \]
Assuming $0 < \theta_0 < 1/3$, we infer that
\[ \int_{\theta_0}^{1} \sqrt{g(v)} (1 - v) \, dv \geq \int_{\theta_0}^{1} \sqrt{v} (1 - v) \, dv \geq \int_{\theta_0}^{1} \sqrt{v} (1 - v) \, dv =: K_1^2. \]
The above arguments immediately imply (2.32) which leads, when applied to (2.31), to
\[ 2\sqrt{K_1 \beta \tau} \leq \frac{c}{2} + \frac{\theta_0}{a} + |c|\theta_0. \]
Again, as $0 < \theta_0 < 1/3$, we obtain
\[ 2\sqrt{K_1 \beta \tau} - \frac{\theta_0}{a} \leq \frac{c}{2} + \frac{|c|}{3}. \tag{2.33} \]
Now, let $K = 12\sqrt{K_1 \beta}$. We distinguish the two cases for $c$. If $c \leq 0$, inequality (2.33) yields
\[ \frac{K \sqrt{\tau}}{6} - \frac{\theta_0}{a} \leq \frac{c}{6} \implies c \geq \Gamma. \]
If $c \geq 0$, inequality (2.33) yields
\[ \frac{K \sqrt{\tau}}{6} - \frac{\theta_0}{a} \leq \frac{5c}{6} \implies c \geq \Gamma / 5. \]
These previous inequalities finally lead to (2.23). The uniform bound in (2.24) directly follows from (2.18), (2.28), and (2.31).
Remark 2.4. In the case where the homotopy parameter $\tau \neq 0$, then $\Gamma > 0$ for sufficiently large $a$ and therefore
\[ c \geq \frac{1}{5} \left( K \sqrt{\tau} - \frac{6\theta_0}{a} \right) > 0. \] (2.34)

We move to the proof of Proposition 2.1.

Proof of Proposition 2.1. Let $K_\tau, F_\tau, \Omega$ be as defined in (2.13)–(2.15), respectively. Lemmas 2.2 and 2.3 show the existence of finite constants $M, \zeta, \bar{c}$ such that
\[ F_\tau(\partial \Omega) \neq \emptyset, \quad \forall \tau \in [0, 1]. \]

It remains to show that the degree $\deg(F_1, \Omega, 0)$ in $\bar{\Omega}$ is not zero. However, the homotopy invariance property of the degree implies that
\[ \deg(F_\tau, \Omega, 0) = \deg(F_0, \Omega, 0), \quad \forall \tau \in [0, 1]. \]

When $\tau = 0$, a straightforward computation gives $U$ and $v$ solutions of (2.10)–(2.12). Indeed,
\[ U(x) = U_0^c(x) = \frac{e^{-\frac{x}{2}a} - e^{-\frac{x}{2}a}}{e^{\frac{a}{2}x} - e^{-\frac{a}{2}x}} \quad \text{and} \quad V(x) = V_0^c(x) = 0. \]
Therefore, $F_0$ is given explicitly as
\[ F_0(u, v, c) = (u - U_0^c, v, \max_{x \geq 0} u - \theta_0). \]
This mapping is homotopic to
\[ \Phi(u, v, c) = (u - U_0^c, v, \max_{x \geq 0} U_0^c - \theta_0), \]
which in turn is homotopic to
\[ \tilde{\Phi}(u, v, c) = (u - U_0^c, v, U_0^c(0) - \theta_0), \]
where $c_0^c$ is the unique number chosen so that $U_0^c(0) = \theta_0$. Using the multiplicative degree property, we know that the degree of $\Phi$ is the product of the degrees of each component. The degree of the first two components is 1, while the degree of the last one is $-1$ as the function $U_0^c(0)$ is decreasing in $c$. Thus the degree of $\tilde{\Phi}$ is $-1$, and therefore the degree of $F_0$ is also $-1$. Consequently, we get $\deg(F_1, \Omega, 0) = -1$ and so $\ker(I - K_1) \neq \emptyset$. This terminates the proof of Proposition 2.1. ■

III. EXISTENCE ON $\mathbb{R}$

In this section, we let $a \to +\infty$ in order to construct, using the $a$ priori estimates of Sec. II, a solution with a positive cutoff $\theta_0$ and hence proving Proposition 1.3. Note that, in all what follows, the homotopy parameter $\tau = 1$.

Proof of Proposition 1.3. Let us consider the solution $(u_a, v_a, c_a)$ of Proposition 2.1. We aim to let $a \to \infty$ and show that $(u_a, v_a, c_a)$ converges to a solution $(u(x; \theta_0), v(x; \theta_0), c(\theta_0))$ of Proposition 1.3. The proof is divided into several steps.

Step 1 (Local convergence). Using (2.18), (2.23), and (2.34), there exists two constants $\zeta$ and $\bar{c}$, independent of $a$ (when $a$ is sufficiently large) with $0 < \zeta < \bar{c} < +\infty$ and such that
\[ \zeta \leq c_a \leq \bar{c}. \]
This implies that the existence of a subsequence $a_n \to +\infty$ so that $c_n = c_{a_n}$ converges to a limit $c_* = c_*(\theta_0)$.

Using (2.17), we can easily see that $\|v_a\|_\infty$ and $\|v'_a\|_\infty$ are bounded independently of $a$. The second equation of (2.1) can be written as
\[ v_a'' = \frac{1}{\varepsilon}(v_a - \lambda u_a), \] (3.1)
and hence [again from (2.17)] we also get that \( \|v''_a\|_\infty \) is bounded independently of \( a \). Therefore \( v_a \) is bounded independently of \( a \) in \( W^{2,\infty}(-a, a) \). As a consequence, we obtain the convergence of \( v_n = v_{a_n} \) to a function \( v \) in the topology of \( C^{1}_{\text{loc}}(\mathbb{R}) \). Set

\[
\phi_a = v_a - \lambda u_a.
\]

We claim that \( \phi_a \) is bounded in \( H^2(-a, a) \) independently of \( a \). The convolution representation of \( v_a \) in (2.3) gives \( v''_a = \lambda K'_x * \bar{u}'_a \), hence, by (1.13),

\[
\|v''_a\| \leq \lambda \|K'_x\|_1 \|\bar{u}'_a\|_2 \leq \lambda \|K'_x\|_1 \|u'_a\|_2 \leq C,
\]

which, in observance of (3.1), shows that \( \phi_a = v''_a \) is bounded in \( L^2(-a, a) \) independently of \( a \). The uniform \( L^2 \) bounds (1.13) on \( v_a \) and \( u'_a \) lead to the same conclusion for \( \phi_a = v''_a - \lambda u''_a \).

We write again the first equation of (2.1) as

\[
-\delta u''_a = cu''_a + g(u_a)u_a(1 - u_a) + (1 - \delta)g'(u_a)u_a u'_a v'_a + (1 - \delta)g(u_a)u_a u'_a v''_a + (1 - \delta)g(u_a)u_a v''_a.
\]

We may claim that \( v''_{\text{loc}} \) is bounded in \( L^{\infty}(-a, a) \) independently of \( a \). Moreover, from all the previous arguments, one can easily deduce that all the remaining terms on the right-hand side of the above inequality are bounded in \( L^2(-a, a) \) independently of \( a \). The same holds then for \( u''_a \) and eventually for \( \phi''_a = v''_a - \lambda u''_a \). We may thus infer that our claim is true and that \( \phi_a \) is bounded in \( H^2(-a, a) \) independently of \( a \). As a consequence, we obtain the convergence of \( \phi_a = \phi_{a_n} \) to a function \( v \) in the topology of \( C^{1}_{\text{loc}}(\mathbb{R}) \).

Accordingly, as \( u_a = 1/(\lambda(v_a - \phi_a)) \), we also obtain the convergence of \( u_{n} = u_{a_n} \) to a function \( u \) in the topology of \( C^{1}_{\text{loc}} \times C^{1}_{\text{loc}} \times \mathbb{R} \), of the sequence \((u_n, v_n, c_n)\) to \((u, v, c)\), satisfying (we drop the dependence on \( \theta_0 \) in the notation)

\[
\begin{align*}
-c, u' - \delta u'' - (1 - \delta)g(u)uv' = g(u)u(1 - u), \\
-\varepsilon v'' + v = \lambda u,
\end{align*}
\]

and

\[
u = \lambda K_x * u.
\]

Furthermore, the lower bound in (2.34) proves that \( c_\varepsilon \geq K/5 > 0 \).

**Step 2 (Boundary conditions).** We show here that \( u \) and \( v \) verify the boundary conditions (1.10). Indeed, the representation (3.3) shows that it is sufficient to prove the boundary conditions only for \( u \). From Step 1, we can easily deduce that the limits of \( u \) at \( \pm \infty \) both exist. Set

\[
u_l = \lim_{x \to -\infty} u(x) \quad \text{and} \quad \nu_r = \lim_{x \to +\infty} u(x).
\]

Since \( g(u) = 0 \) for \( u \leq \theta_0 \), the function \( u \) can be explicitly computed for \( x \geq 0 \) as follows:

\[
u(x) = \theta_0 e^{-\frac{\varepsilon}{2} x}, \quad \forall x \geq 0.
\]

Therefore, as \( c_\varepsilon > 0 \), we deduce that \( \nu_l = 0 \). It remains to show that \( \nu_l = 1 \) for \( \theta_0 \) small enough. Again, as \( g(u) = 0 \) for \( u \leq \theta_0 \), we deduce by the maximum principle that

\[
u(x) \geq \theta_0, \quad \forall x \leq 0,
\]

and thus \( \nu_l \geq \theta_0 \). However, from the uniform bound of Proposition 2.1,

\[
\int_{-a}^{a} g(u)u_a(1 - u_a)^2 \, dx \leq K,
\]

we deduce that

\[
\int_{-\infty}^{\infty} g(u)u(1 - u)^2 \, dx \leq K,
\]

hence either \( \nu_l = 1 \) or \( \nu_l = \theta_0 \). To get a contradiction, we assume \( \nu_l = \theta_0 \) to be small enough. Integrating the first equation of (3.2), we get

\[
c_\varepsilon \theta_0 = \int_{-\infty}^{\infty} g(u)u(1 - u) \, dx.
\]
Multiplying the same equation by \( u \) and integrating give
\[
\frac{c_s \theta_0^2}{2} + \delta \int_{-\infty}^{\infty} |u'|^2 \, dx + (1 - \delta) \int_{-\infty}^{\infty} g(u)uu' \, dx = \int_{-\infty}^{\infty} g(u)u^2(1 - u) \, dx
\]
where we used (3.5) for the second equality. By (3.3), we have the estimate \( ||u'||_2 \leq \lambda ||u'||_2 \); therefore, using Hölder’s inequality, we estimate
\[
\left| \int_{-\infty}^{\infty} g(u)uu' \, dx \right| \leq ||u'||_2 ||u'||_2 \leq \lambda \int_{-\infty}^{\infty} |u'|^2 \, dx.
\]
Using this inequality in (3.6), we deduce that
\[
\frac{c_s \theta_0^2}{2} + \beta \int_{-\infty}^{\infty} |u'|^2 \, dx + \int_{-\infty}^{\infty} g(u)u(1 - u)^2 \, dx \leq c_s \theta_0,
\]
where \( \beta \) is given by (2.30). Since \( u_l = u(0) = \theta_0 \) and \( u(x) \geq \theta_0 \) for \( x \leq 0 \), then the function \( u \) attains its maximum at some point \( x_M \leq 0 \) with, by (3.5), \( u_M = u(x_M) > \theta_0 \). We claim that \( u_M < 1/2 \) for \( \theta_0 \) small enough. In fact, if \( u_M \geq 1/2 \) and \( u_l = \theta_0 < 1/3 \), then there exists a constant \( C_1 > 0 \) independent of \( \theta_0 \) such that
\[
\int_{-\infty}^{\infty} |u'|^2 \, dx + \int_{-\infty}^{\infty} g(u)u(1 - u)^2 \, dx \geq C_1.
\]
Therefore, as \( c_s \) is bounded from above [see (2.18)], we can see that choosing \( \theta_0 \) small enough violates inequality (3.7).

Again, integrate the first equation of (3.2) but from \( -\infty \) to \( x_M \) to obtain, with \( \theta_0 < u_M < 1/2, \)
\[
-c_s(u_M - \theta_0) - (1 - \delta)g(u_M)u_Mu'(u_M) = \int_{x_0}^{x_M} g(u)u(1 - u) \, dx \geq 0.
\]
Remark that for the cutoff \( g_0 \), there exists a constant \( C_2 > 0 \) such that
\[
g_0(\sigma) \leq C_2(\sigma - 1), \quad \forall \sigma \geq 1,
\]
hence, for \( u \geq \theta_0 \), we have
\[
g(u) = g_0 \left( \frac{u}{\theta_0} \right) \leq C_2 \left( \frac{u}{\theta_0} - 1 \right).
\]
Also remark, from (3.3), that \( ||u'||_\infty \leq C ||u||_\infty = Cu_M \). The above arguments with (3.8) lead to
\[
-c_s\theta_0(u_M - \theta_0) + C u_M^2 (u_M - \theta_0) \geq 0.
\]
Since \( c_s > 0 \) and \( u_M > \theta_0 \), we have
\[
u_M^2 \geq K \theta_0
\]
and, in particular, \( \theta_0 \leq \frac{u_M}{\sqrt{2}} \) if \( \theta_0 \) is sufficiently small. As \( u_l = \theta_0 \), let \( x_0 \) be the first point to the left of \( x_M \) such that \( u(x_0) = \frac{u_M}{\sqrt{2}} \). Therefore \( u(x) \in \left[ \frac{u_M}{\sqrt{2}}, u_M \right] \) for all \( x \in [x_0, x_M] \) and \( g(u(x)) = 1 \) on this interval.

Letting \( L = x_M - x_0 \), we deduce, using (3.7), that
\[
c_s \theta_0 \geq \beta \int_{x_0}^{x_M} |u'|^2 \, dx + \int_{x_0}^{x_M} g(u)u(1 - u)^2 \, dx \geq C \left( \frac{u_M^2}{L} + u_M L \right) \geq C u_M^3/2,
\]
where it follows that \( u_M \leq C \theta_0^{2/3} \), which contradicts (3.9). Consequently \( u_l = 1 \) and this terminates the proof of Proposition 1.3.

IV. PROOF OF THEOREMS 1.1 AND 1.2

In this section, we remove the cutoff \( g \) by letting \( \theta_0 \to 0 \) and thus proving Theorems 1.1 and 1.2. We notice that the solutions \( (u(x; \theta_0), v(x; \theta_0), c(\theta_0)) \) of Proposition 1.3 are invariant by translation with the left and right limits (1.10). Hence, we may translate \( u(x; \theta_0) \) and \( v(x; \theta_0) \) so that we fix \( u(0; \theta_0) = 1/2 \). We now show the following.
Proof of Theorem 1.2. As in the case $a \to \infty$, the uniform bounds in Proposition 1.3 allow us to find a sequence $\theta_{0,n} \to 0$ where the speeds $c_n = c(\theta_{0,n})$ converge to $c_1 > 0$, and the functions $u_n = u(x; \theta_{0,n})$ and $v_n = v(x; \theta_{0,n})$ converge to $u(x)$ and $v(x)$, respectively. Note that as $\theta_{0,n} \to 0$, then the cutoff $g(u_n)$ converges to $\Psi$ with $\Psi(x) = 1$ on $\{x; u(x) \neq 0\}$. The functions $u$ and $v$ satisfy
\[
\begin{cases}
-c_1 u' - \delta u'' - (1 - \delta)(\Psi(x)u'v') = \Psi(x)u(1-u), \\
-\varepsilon v'' + v = \lambda u.
\end{cases}
\]
Using the elliptic regularity, in particular that $u \to g(u)u$ is Lipschitz continuous, we can immediately see that $u_n$ and $v_n$ are uniformly bounded in $C^{2,\alpha}(\mathbb{R})$ and thus the limits $u$ and $v$. Using this and the maximum principle, we finally conclude that $u > 0$ and thus $\Psi = 1$. The functions $u$ and $v$ eventually satisfy
\[
\begin{cases}
-c_1 u' - \delta u'' - (1 - \delta)(uv') = u(1-u), \\
-\varepsilon v'' + v = \lambda u,
\end{cases}
\]
where we again have the representation
\[
v = \lambda K_G + u.\tag{4.2}
\]
It remains to show the boundary conditions (1.10), where, once again, it is sufficient to show the boundary condition only for $u$ where we conclude that of $v$ by (4.2). We argue as in Step 2 of the proof of Proposition 1.3. First, the existence of the limits at $\pm \infty$ follows from the $L^2$ bounds of the gradient of $u$ and standard elliptic regularity estimates. When passing to the limit $\theta_{0,n} \to 0$, the estimate (3.4) now reads
\[
\int_{-\infty}^{\infty} u(1-u)^2 dx \leq K.
\]
Therefore, the only possible values for $u_l$ and $u_r$ are 0 and 1, respectively. To finish the proof, we show that $u_l > u_r$. By integrating the first equation of (4.3) over $\mathbb{R}$, we get
\[
c_1 (u_l - u_r) = \int_{-\infty}^{\infty} u(1-u) dx.
\]
Therefore, multiplying the same equation by $u$ and integrating lead to
\[
\frac{c_1 (u_l^2 - u_l^2)}{2} + \delta \int_{-\infty}^{\infty} |u'|^2 dx + (1 - \delta) \int_{-\infty}^{\infty} uu' dx = \int_{-\infty}^{\infty} u^2 (1-u) dx = c_1 (u_l - u_r) - \int_{-\infty}^{\infty} u(1-u)^2 dx.
\]
Using Hölder's inequality and (4.2), we estimate
\[
\left| \int_{-\infty}^{\infty} uu' dx \right| \leq ||u'||_2 ||v'||_2 \leq \lambda \int_{-\infty}^{\infty} |u'|^2 dx,
\]
leading to
\[
\frac{c_1 (u_l^2 - u_l^2)}{2} + \int_{-\infty}^{\infty} u(1-u)^2 dx + \beta \int_{-\infty}^{\infty} |u'|^2 dx \leq c_1 (u_l - u_r),
\]
which may be rewritten as
\[
\int_{-\infty}^{\infty} u(1-u)^2 dx + \beta \int_{-\infty}^{\infty} |u'|^2 dx \leq c_1 (u_l - u_r) \left( 1 - \frac{u_l + u_r}{2} \right).
\]
Since $u(0)(1-u(0))^2 = \frac{1}{8} > 0$, then the left side of the above inequality is positive. However, as $u_l + u_r \leq 2$, we deduce that $\left( 1 - \frac{u_l + u_r}{2} \right) \geq 0$. It directly follows that $u_l > u_r$ and the proof is complete.

We are now ready to present the following.

Proof of Theorem 1.1. By Theorem 1.1, we know that $u, v \in C^{2,\alpha}(\mathbb{R})$ whence, by the second equation of (4.3), we can easily see that $v \in C^{4,\alpha}(\mathbb{R})$. Setting
\[
w = -v',
\]
we obtain \( w \in C^3(\mathbb{R}) \) and \( w(\pm \infty) = 0 \). Differentiating the second equation of (4.3) and using the above expression of \( w \) in (4.3), we finally deduce that the triplet \((u, w, c)\) satisfies

\[
\begin{aligned}
-\epsilon w'' + w &= -\lambda u', \\
-\epsilon^2 w''' + (1 - \delta)(uw)' &= u(1 - u),
\end{aligned}
\]

leading to the proof of Theorem 1.1.

\[\blacksquare\]

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