A blow-up result for a nonlinear damped wave equation in exterior domain: The critical case

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We consider the initial boundary value problem of the nonlinear damped wave equation in an exterior domain $\Omega$. We prove a blow-up result which generalizes the result of non-existence of global solutions of Ogawa and Takeda (2009). We also show that the critical exponent belongs to the blow-up case.

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\textbf{1. Introduction}

This paper concerns the initial boundary value problem of the nonlinear damped wave equation in an exterior domain. Let $\Omega \subset \mathbb{R}^n$ be an exterior domain whose obstacle $\partial \subseteq \mathbb{R}^n$ is bounded with smooth compact boundary $\partial \Omega$. We consider the initial boundary value problem

\begin{equation}
\begin{cases}
\partial_t^2 u - \Delta u + \partial_t u = |u|^p & t > 0, x \in \Omega, \\
u(0, x) = u_0(x), & x \in \Omega, \\
\partial_t u(0, x) = u_1(x) & t > 0, x \in \partial \Omega,
\end{cases}
\end{equation}

where the unknown function $u$ is real-valued, $n \geq 1$ and $p > 1$. Throughout this paper, we assume that

\begin{equation}
(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)
\end{equation}

and

\begin{equation}
\text{supp} u_0 \cup \text{supp} u_1 \text{ is compact in } \Omega.
\end{equation}

To begin with, let us make mention the Cauchy problem in $\mathbb{R}^n$. Matsumura [1] showed that the existence of the global solutions of (1.1), under the assumption $n = 1$, $p > 3$ and $n = 2$, $p > 2$ for the small data. Kawashima–Nakao–Ono [2] proved the existence of a global solution with small initial data in the case $1 + 4/n \leq p < 1 + 4/(n - 2)$ (see also Nakao–Ono [3]). Todorova–Yordanov [4] showed the exponent $p = 1 + 2/n$ is the critical one for (1.1), which classifies the global existence of solutions and finite time blow-up for the small data. The exponent $p = 1 + 2/n$ is known as the critical situation for the results of the semilinear heat equation. Namely, if $p > 1 + 2/n$, the solution exists globally in time, and if
1 < p ≤ 1 + 2/n and the initial data is small, then the solution blows up in a finite time. The critical exponent p = 1 + 2/n is called the Fujita exponent due to the work of Fujita [5]. For 1 < p < 1 + 2/n, we refer the reader to [6,4,7,8] for various blowing up results. Li–Zhou [6] showed the finite time blow-up of the solution for the subcritical case 1 < p ≤ 1 + 2/n and they obtained the upper bound of the lifespan of solutions for n = 1, 2. Todorova–Yordanov [4] proved that when 1 < p < 1 + 2/n and n ≥ 1, the solution u blows up in a finite time. Zhang [7] extends the blow-up result to the critical case p = 1 + 2/n by using an approach which is not only much shorter but also capable of proving more general blow-up results than those in [4].

Concerning our problem (1.1), Ikehata [9,10] proved that the solution exists globally when n = 2 for p > 1 + 2/n, and n = 3, 4, 5 for 1 + 4/(n + 2) < p ≤ 1 + 2/(n − 2) under the assumption of the compactness of the initial data. Ono [11] showed the same result of global existence of solution without compactness of the initial data, applying the result of Dan–Shibata [12] and cut-off method. Ogawa–Takeda [13] proved the non-existence of non-negative global weak solutions of (1.1) under the assumption that u0 = 0, 1 < p < 1 + 2/n, and the support of the initial data is compact. However, in this case, the blow-up of the solutions remains an open question. In addition, up to our knowledge, the blow-up solutions of (1.1) in the critical case (p = 1 + 2/n) have not been treated in the literature.

Before stating our main theorem, we introduce some notations. For R > 0, let $B_R(x_0) := \{x \in \mathbb{R}^n; |x - x_0| < R\}$. We introduce the first eigenfunction $\varphi_R$ of $-\Delta$ and the first eigenvalue $\lambda_1$ over $\Omega_R$:

$$
\begin{align*}
-\Delta \varphi_R &= \lambda_1 \varphi_R \quad \text{in } \Omega_R, \\
\varphi_R > 0 &\quad \text{in } \Omega_R, \\
\varphi_R &= 0 \quad \text{in } \partial \Omega_R, \\
\|\varphi_R\|_{L^\infty(\Omega_R)} &= 1.
\end{align*}
$$

The aim of this paper is to generalize the results of Ogawa–Takeda [13] by proving the blow-up of solutions of (1.1) under weaker assumptions on the initial data. Moreover, we extend this result to the critical case $p = 1 + 2/n$. Our results are based on the test function method introduced by Zhang [7] and modified by Fino–Karch [14]. Namely

**Theorem 1.1.** Let $n \geq 1$, 1 < $p \leq 1 + 2/n$, and $\text{suppu}_0 \cup \text{suppu}_1$ is compact. Assume, for arbitrary $0 < \epsilon_0 < 1$ and for large $R > 0$, that the initial data satisfy

$$
\text{suppu}_0 \cup \text{suppu}_1 \subset \{x \in \Omega_R; \varphi_R > \epsilon_0\}
$$

and

$$
\int_{\Omega} u_0(x) \, dx \geq 0, \quad \int_{\Omega} u_1(x) \, dx > 0.
$$

Then the mild solution of the problem (1.1) blows up in finite time.

This paper is organized as follows: in Section 2, we present some definitions and properties that are useful in the sequel. Section 3 is devoted to the proof of the blow-up result (Theorem 1.1).

## 2. Preliminaries

In this section, we introduce some notations and basic results. Using Lemma 3 in [13], there exist $C_1, C_2 > 0$ independent of $R$ such that

$$
C_1 R^{-2} \leq \lambda_1 \leq C_2 R^{-2},
$$

where $\lambda_1$ is the first eigenvalue of $\varphi_R$ introduced in Section 1.

Now, we consider the homogeneous equation corresponding to (1.1)

$$
\begin{align*}
\partial_t^2 u - \Delta u + \partial_t u &= 0 \quad t > 0, \, x \in \Omega, \\
u(0, x) &= 0, \quad \partial_t u(0, x) = g(x) \quad x \in \Omega, \\
0 &= 0, \quad t > 0, \, x \in \partial \Omega.
\end{align*}
$$

By Proposition 8.4.2 in [15], this problem has a unique global solution denoted by $S(t)g$. We now give the definition of mild solutions.

**Definition 2.1 (Mild Solution).** Let $T > 0$. We say that $u$ is a mild solution if $u \in C([0, T], H^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ and satisfies

$$
u(t) = \partial_t S(t)u_0 + S(t)(u_0 + u_1) + \int_0^t S(t - s)u_1^p(s) \, ds.
$$
**Proposition 2.2** ([15, Proposition 8.4.2]). Let $1 < p \leq n/(n - 2)$ for $n \geq 3$, and $p \in (1, \infty)$ for $n = 1, 2$. Under the assumption (1.2), there exists a maximal existence time $0 < T_{\text{max}} \leq \infty$ such that the problem (1.1) has a unique mild solution $u \in C([0, T_{\text{max}}], H^1_0(\Omega)) \cap C^1([0, T_{\text{max}}], L^2(\Omega))$. Moreover, we have the following alternative

\[ \text{either } T_{\text{max}} = \infty \text{ or else } T_{\text{max}} < \infty \text{ and } \|u(t)\|_{H^1} + \|u(t)\|_2 \to \infty \text{ as } t \to T_{\text{max}}. \tag{2.3} \]

**Remark 2.3.** We say that $u$ is a global solution of (1.1) if $T_{\text{max}} = \infty$, while in the case of $T_{\text{max}} < \infty$, we say that $u$ blows up in finite time.

**Definition 2.4** (Weak Solution). Let $T > 0$, $p > 1$, and $u_0, u_1 \in L^1_{\text{loc}}(\Omega)$. We say that $u$ is a weak solution of (1.1) if $u \in L^p((0, T), L^2_{\text{loc}}(\Omega))$ and satisfies

\[
\int_0^T \int_{\Omega} |u|^p \varphi \, dx \, dt + \int_\Omega (u_0(x) + u_1(x)) \varphi(0, x) \, dx - \int_{\Omega} u_0(x) \varphi_t(0, x) \, dx 
= \int_0^T \int_{\Omega} u \varphi_{tt} \, dx \, dt - \int_0^T \int_{\Omega} u \varphi_t \, dx \, dt - \int_0^T \int_{\Omega} u \Delta \varphi \, dx \, dt, \quad \tag{2.4}
\]

for all compactly supported function $\varphi \in C^2([0, T] \times \Omega)$ such that $\varphi(\cdot, T) = 0$ and $\varphi_t(\cdot, T) = 0$.

Next, the following lemma is useful for the proof of Theorem 1.1. The proof of this lemma is much the same procedure as in the proof of [16, Lemma 4.2].

**Lemma 2.5** (Mild $\rightarrow$ Weak). Let $T > 0$. Suppose that $1 < p \leq n/(n - 2)$, if $n \geq 3$, and $p \in (1, \infty)$, if $n = 1, 2$. If $u \in C([0, T], H^1_0(\Omega)) \cap C^1([0, T], L^2(\Omega))$ is the mild solution of (1.1), then $u$ is a weak solution of (1.1).

**Remark 2.6.** The notion of mild solutions is crucial in the proof of Theorem 1.1. In fact, with this notion we have the alternative (2.3) leading to our blow-up result.

Throughout this paper, positive constants will be denoted by $C$ and will change from line to line.

### 3. Blow-up result

In this section we devote ourselves to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** We argue by contradiction assuming that $u$ is not a blow-up solution of (1.1). Then, using (2.3), $u$ is a global mild solution of (1.1). From Lemma 2.5 we have

\[
\int_0^T \int_{\Omega} |u|^p \varphi \, dx \, dt + \int_\Omega (u_0(x) + u_1(x)) \varphi(0, x) \, dx - \int_{\Omega} u_0(x) \varphi_t(0, x) \, dx 
= \int_0^T \int_{\Omega} u \varphi_{tt} \, dx \, dt - \int_0^T \int_{\Omega} u \varphi_t \, dx \, dt - \int_0^T \int_{\Omega} u \Delta \varphi \, dx \, dt, \quad \tag{3.1}
\]

for all $T > 0$ and all compactly supported function $\varphi \in C^2([0, T] \times \Omega)$ such that $\varphi(\cdot, T) = 0$ and $\varphi_t(\cdot, T) = 0$. Since $\vartriangle$ is compact, there exists $r_0 > 0$ such that

\[
\vartriangle \cup \text{supp } u_0 \cup \text{supp } u_1 \subset B_{r_0}(0).
\]

Let $R > r_0$, and let $B > 0$ be a constant that will be determined later. Take $\varphi(x, t) = \varphi_B(x)\eta_B^{2p'}(t)$ where $\eta_B(t) := \eta\left(\frac{t}{B}\right)$ and $\eta(\cdot) \in C^\infty(\mathbb{R}_+)$ is a cut-off non-increasing function such that

\[
\eta(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 1/2, \\
0 & \text{if } t \geq 1,
\end{cases}
\]

$0 \leq \eta(t) \leq 1$ and $|\eta'(t)| \leq C$ for some $C > 0$ and all $t > 0$. We obtain

\[
\int_0^R \int_{\Omega_R} |u|^p \varphi_B(x)\eta_B^{2p'}(t) \, dx \, dt + \int_{\Omega_R} (u_0(x) + u_1(x)) \varphi_B(x) \, dx 
= \int_0^R \int_{\Omega_R} u \varphi_B(x)\Delta^2[\eta_B^{2p'}(t)] \, dx \, dt - \int_0^R \int_{\Omega_R} u \varphi_B(x)\Delta[\eta_B^{2p'}(t)] \, dx \, dt
- \int_0^R \int_{\Omega_R} u \eta_B^{2p'}(t) \Delta \varphi_B(x) \, dx \, dt. \quad \tag{3.2}
\]
From (1.5) and (1.6), we have

\[
\int_{\Omega_R} (u_0(x) + u_1(x)) \varphi_R(x) \, dx \geq \int_{x \in \Omega_R; \varphi_R > \epsilon_0} (u_0(x) + u_1(x)) \varphi_R(x) \, dx
\]

\[
> \epsilon_0 \int_{x \in \Omega_R; \varphi_R > \epsilon_0} (u_0(x) + u_1(x)) \, dx > 0.
\]

(3.3)

Therefore

\[
\int_0^B \int_{\Omega_R} |u|^p \varphi_R(x) \eta_B^{2p'}(t) \, dx \, dt
\]

\[
\leq \int_0^B \int_{\Omega_R} u \varphi_R(x) \partial_t^2 \eta_B^{2p'}(t) \, dx \, dt - \int_0^B \int_{\Omega_R} u \varphi_R(x) \partial_t \eta_B^{2p'}(t) \, dx \, dt
\]

\[
+ \int_0^B \int_{\Omega_R} u \eta_B^{2p'} (t) (-\Delta) \varphi_R(x) \, dx \, dt
\]

\[
=: I_1 - I_2 + I_3.
\]

(3.4)

Firstly, to make this paper self-contained, we estimate the left-hand side of (3.4) by using the same steps that have been adopted in [13, Lemma 4]. Set \( F(t) := \int_{x \in \Omega_R; \varphi_R > \epsilon_0} u(t, x) \, dx \). By assumptions (1.5) and (1.6), we see that \( F(0) \geq 0 \) and \( F'(0) > 0 \). Thus \( F(t) \) is monotone increasing and \( F(t) > 0 \) in \([0, T_1]\) for some time \( T_1 > 0 \). Therefore the following property

\[
|\text{supp } u(t) \cap \{x; \varphi_R > \epsilon_0\}| > 0, \quad t \in [0, T_1]
\]

(3.5)

holds. Observing that \( \text{supp } u_0 \) and \( \text{supp } u_1 \) are compact in \( \Omega \) and \( \{x; \varphi_R > \epsilon_0\} \) is a bounded open set in \( \Omega \), we obtain

\[
d := \text{dist} (\text{supp } u_0 \cup \text{supp } u_1, \ \text{the complement of } \{x; \varphi_R > \epsilon_0\}) > 0.
\]

Hence there exists sufficiently small \( T_2 \in (0, T_1) \) such that

\[
\text{supp } u(t) \subset \{x \in \Omega_R; \varphi_R > \epsilon_0\}
\]

for arbitrary \( t \in [0, T_2] \). This gives

\[
|\text{supp } u(t) \cap \{x; \varphi_R > \epsilon_0\}| = |\text{supp } u(t)|
\]

(3.7)

for arbitrary \( t \in [0, T_2] \). As a conclusion, for \( t \in [0, T_2] \), we have from (3.5), (3.6), and (3.7) that

\[
T_2 < \frac{B}{2}, \quad \text{supp } u(t) \subset \{x; \varphi_R > \epsilon_0\}, \quad \text{and } |\text{supp } u(t)| > 0.
\]

Therefore

\[
\int_0^B \int_{\Omega_R} |u|^p \varphi_R(x) \eta_B^{2p'}(t) \, dx \, dt \geq \int_0^B/2 \int_{\varphi_R > \epsilon_0} |u|^p \, dx \, dt
\]

\[
\geq \epsilon_0 \int_0^{T_2/2} \int_{\varphi_R > \epsilon_0} |u|^p \, dx \, dt
\]

\[
\geq \epsilon_0 \int_0^{B/2} \int_{\text{supp } u(t)} |u|^p \, dx \, dt > 0
\]

\[
=: M > 0.
\]

(3.8)

On the other hand, to estimate the right-hand side of (3.4), we introduce the term \( \varphi^{1/p} \varphi^{-1/p} \) in \( I_1, I_2 \) and \( I_3 \), and we use Young’s inequality to obtain

\[
I_1 \leq \int_0^B \int_{\Omega_R} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_R(x) \left| \partial_t^2 \eta_B^{2p'}(t) \right| \, dx \, dt
\]

\[
\leq \frac{1}{6} \int_0^B \int_{\Omega_R} |u|^p \varphi \, dx \, dt + C \int_0^B \int_{\Omega_R} \varphi^{-p'/p} \varphi_R(x) \left| \partial_t \eta_B^{2p'}(t) \right|^{p'} \, dx \, dt
\]

\[
\leq \frac{1}{6} \int_0^B \int_{\Omega_R} |u|^p \varphi \, dx \, dt + C \int_0^B \int_{\Omega_R} \varphi_R(x) \left| \partial_t \eta_B \right|^{2p'} \, dx \, dt
\]

\[
+ C \int_0^B \int_{\Omega_R} \varphi_R(x) \eta_B^{2p'}(t) \left| \partial_t^2 \eta_B \right|^{p'} \, dx \, dt.
\]

(3.9)
Using the definition of the limit, there exists

\[ I_2 \leq \int_0^B \int_{\Omega_R} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_R(x) \left| \frac{\partial_t [\eta^p_R(t)]}{\partial_t [\eta^p_R(t)]} \right| \, dx \, dt \]

\[ \leq \frac{1}{6} \int_0^B \int_{\Omega_R} |u|^p \varphi \, dx \, dt + C \int_0^B \int_{\Omega_R} \varphi^{-p/2} \varphi_R \left( \varphi \right) \left| \frac{\partial_t [\eta^p_R(t)]}{\partial_t [\eta^p_R(t)]} \right| \, dx \, dt \]

\[ = \frac{1}{6} \int_0^B \int_{\Omega_R} |u|^p \varphi \, dx \, dt + C \int_0^B \int_{\Omega_R} \varphi_R(x) \eta^p_R(t) \left| \frac{\partial_t \eta_R}{\partial_t \eta_R} \right| \, dx \, dt. \quad (3.10) \]

\[ I_3 = \int_0^B \int_{\Omega_R} u \eta^p_R \left( t \left( -\Delta \right) \varphi_R(x) \right) \, dx \, dt \]

\[ = \lambda_1 \int_0^B \int_{\Omega_R} u \eta^p_R \left( t \varphi_R(x) \right) \, dx \, dt \]

\[ \leq C R^{-2} \int_0^B \int_{\Omega_R} |u| \eta^p_R \left( t \varphi_R(x) \right) \, dx \, dt \]

\[ = \int_0^B \int_{\Omega_R} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_R \left( C R^{-2} \eta^p_R \left( t \varphi_R(x) \right) \right) \, dx \, dt \]

\[ \leq \frac{1}{6} \int_0^B \int_{\Omega_R} |u|^p \varphi \, dx \, dt + C R^{-2p} \int_0^B \int_{\Omega_R} \varphi^{-p/2} \varphi_R \left( \eta^p_R \left( t \varphi_R(x) \right) \right) \, dx \, dt \]

\[ = \frac{1}{6} \int_0^B \int_{\Omega_R} |u|^p \varphi \, dx \, dt + C R^{-2p} \int_0^B \int_{\Omega_R} \eta^p_R \left( t \varphi_R(x) \right) \, dx \, dt. \quad (3.11) \]

Using (3.8)–(3.11), it follows from (3.4) that

\[ 0 < M \leq \int_0^B \int_{\Omega_R} |u|^p \varphi \, dx \, dt \]

\[ \leq C \int_0^B \int_{\Omega_R} \varphi_R(x) \left| \frac{\partial_t \eta_R}{\partial_t \eta_R} \right|^2 \, dx \, dt + C \int_0^B \int_{\Omega_R} \varphi_R(x) \eta^p_R \left( t \right) \left| \frac{\partial_t \eta_R}{\partial_t \eta_R} \right|^p \, dx \, dt \]

\[ + C \int_0^B \int_{\Omega_R} \varphi_R(x) \eta^p_R \left( t \right) \left| \frac{\partial_t \eta_R}{\partial_t \eta_R} \right|^p \, dx \, dt + C R^{-2p} \int_0^B \int_{\Omega_R} \eta^p_R \left( t \varphi_R(x) \right) \, dx \, dt. \quad (3.12) \]

At this stage, we have to distinguish 2 cases.

- **Case 1**: \( 1 < p < 1 + 2/n \). In this case, we take \( B = R^2 \). So, using the change of variables: \( s = R^{-2}t, \ y = R^{-1}x \), we get from (3.12) that

\[ 0 < M \leq C R^{-4p/2 + 2 + n} + C R^{-4p/2 + 2 + n} + C R^{-2p/2 + 2 + n} + C R^{-2p/2 + 2 + n} \]

\[ \leq C R^{-2p/2 + 2 + n} \quad (3.13) \]

where \( C \) is independent of \( R \). By choosing \( R > r_0 \) sufficiently large, say \( R \geq \left( \frac{C}{R_0} \right)^{\frac{1}{2(n-2)}} \), then we get a contradiction.

- **Case 2**: \( p = 1 + 2/n \). In this case, we take \( B = R^2 \frac{T}{R} \) where \( T \) is a positive constant, independent of \( R \), to be determined. Moreover, from Case 1 and the fact that \( p = 1 + 2/n \), there exists a positive constant \( D \) independent of \( R \) and \( T \) such that

\[ \int_0^B \int_{\Omega_R} \varphi \, dx \, dt \leq D, \quad \text{for all } R, T > 0, \]

which implies that

\[ \int_{B/2}^B \int_{\Omega_R} \varphi \, dx \, dt \to 0 \quad \text{as } R \to \infty. \]

Using the definition of the limit, there exists \( A = A(T) > 0 \) such that for all \( R > A \), we have

\[ \int_{B/2}^B \int_{\Omega_R} \varphi \, dx \, dt \leq \frac{1}{T^{2(p+1)}}. \quad (3.14) \]
On the other hand, we use Hölder’s inequality instead of Young’s one in $I_1$ and $I_2$, together with the same change of variables, we get

$$I_1 = \int_{B/2}^{B} \int_{\Omega_R} u \varphi(x) \partial_t^2 [\eta_B^{2p}(t)] \, dx \, dt$$

$$\leq \int_{B/2}^{B} \int_{\Omega_R} |u| \varphi^{\frac{1}{p}} \varphi^{-\frac{1}{p}} \varphi_R(x) \left| \partial_t^2 [\eta_B^{2p}(t)] \right| \, dx \, dt$$

$$\leq \left( \int_{B/2}^{B} \int_{\Omega_R} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p'}} \left( \int_{B/2}^{B} \int_{\Omega_R} \varphi^{-\frac{1}{p}} \varphi_R(x) \left| \partial_t^2 [\eta_B^{2p}(t)] \right|^p \, dx \, dt \right)^{\frac{1}{p'}}$$

$$\leq \left( \int_{B/2}^{B} \int_{\Omega_R} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p'}} \left( C \int_{B/2}^{B} \int_{\Omega_R} \varphi_R(x) \left[ |\partial_t \eta_B(t)|^{2p} + \eta_B^{2p}(t) \right] \, dx \, dt \right)^{\frac{1}{p'}}$$

$$\leq C R^{-4 + \frac{2m}{p}} T^{-\frac{2}{p}} \left( \int_{B/2}^{B} \int_{\Omega_R} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p'}}$$

$$\leq C R^{-4 + \frac{2m}{p}} \leq C R^{-2} \leq C T^{-2},$$

thanks to (3.14), the fact that $p = 1 + 2/n$, and choosing $T < R$. Similarly

$$I_2 \leq C R^{-4 + \frac{2m}{p}} T^{-\frac{2}{p}} \left( \int_{B/2}^{B} \int_{\Omega_R} |u|^p \varphi \, dx \, dt \right)^{\frac{1}{p'}} \leq C T^{-2}.$$  \hspace{1cm} (3.15)

For $I_3$, we use (as in the Case 1) Young’s inequality, the fact that $B = R^2 T^{-2}$ and $p = 1 + 2/n$, to get

$$I_3 \leq \frac{1}{2} \int_{B/2}^{B} \int_{\Omega_R} |u|^p \varphi \, dx \, dt + C R^{-2p + 2/n} T^{-2}$$

$$= \frac{1}{2} \int_{B/2}^{B} \int_{\Omega_R} |u|^p \varphi \, dx \, dt + C T^{-2}.$$  \hspace{1cm} (3.17)

Finally, using (3.8) and (3.15)–(3.17), it follows from (3.4) that

$$0 < M \leq \int_{B/2}^{B} \int_{\Omega_R} |u|^p \varphi \, dx \, dt \leq C T^{-2},$$

hence we get a contradiction by taking $T$ large enough, say $T > \sqrt{\frac{c}{M}}$. This completes the proof of Theorem 1.1. \hfill \Box

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